Course Curriculum

Course Title: Calculus-I **Course Code: ME 1201**

Week	Topics
1	*Functions*
2	*Functions*
3	*Functions*
4	*Limits*
5	*Limits*
6	*Differentiation rules*
7	*Differentiation rules*
8	*The Chain Rule, Implicit Differentiation*
9	*The Chain Rule, Implicit Differentiation*
10	*Applications of differentiation*
11	*Applications of differentiation*
12	*Exponential and logarithmic functions.*
13	*Trigonometric functions and their derivatives*
14	*Derivative and integral functions.*
15	*Derivative and integral functions.*
First Semester Exam	

- Textbooks:

- *Thomas, G. B., Haas, J., Heil, C., & Weir, M. (2018). Thomas' Calculus. Pearson Education Limited.*

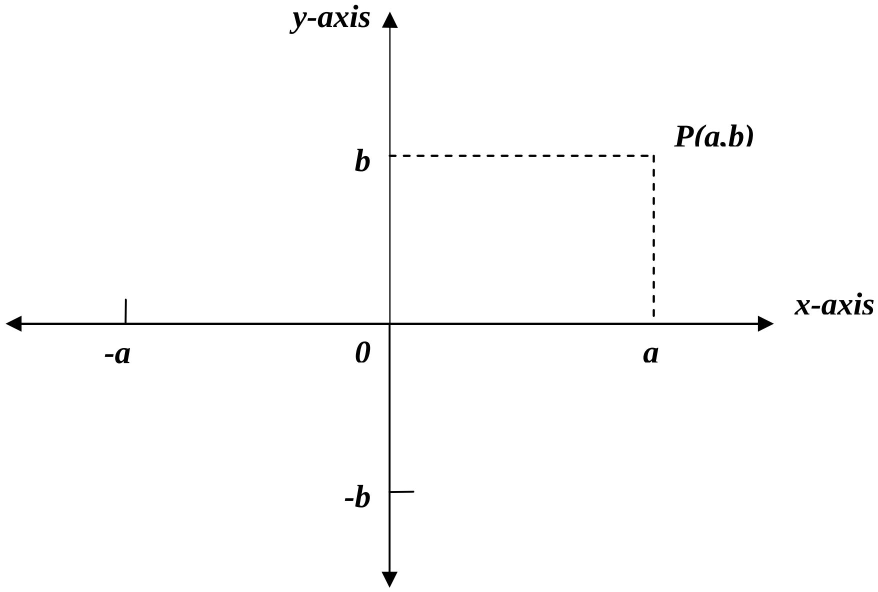
Chapter – one
The Rate of Change of a Function

1-1- Coordinates for the plane :

***Cartesian Coordinate*- Two number lines , one of them horizontal (called *x-axis*) and the other vertical (called *y-axis*). The point where the lines cross is the *origin* . Each line is assumed to represent the real number .**

On the *x-axis* , the positive number a lies a units to the right of the *origin* , and the negative number $-a$ lies a units to the left of the *origin* . On the *y-axis* , the positive number b lies b units above the *origin* while the negative where $-b$ lies b units below the *origin* .

With the axes in place , we assign a pair (a,b) of real number to each point P in the plane . The number a is the number at the foot of the perpendicular from P to the *x-axis* (called *x-coordinate of P*). The number b is the number at the foot of the perpendicular from P to the *y-axis* (called *y-coordinate of P*).



1-2- The Slope of a line :

***Increments* – When a particle moves from one position in the plane to another , the net changes in the particle's coordinates are calculated by subtracting the coordinates of the starting point (x_1, y_1) from the coordinates of the stopping point (x_2, y_2) ,**

i.e. $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$.

***Slopes of nonvertical lines* :**

Let L be a nonvertical line in the plane ,

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on L.

Then the slope m is :

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad where \quad \Delta x \neq 0$$

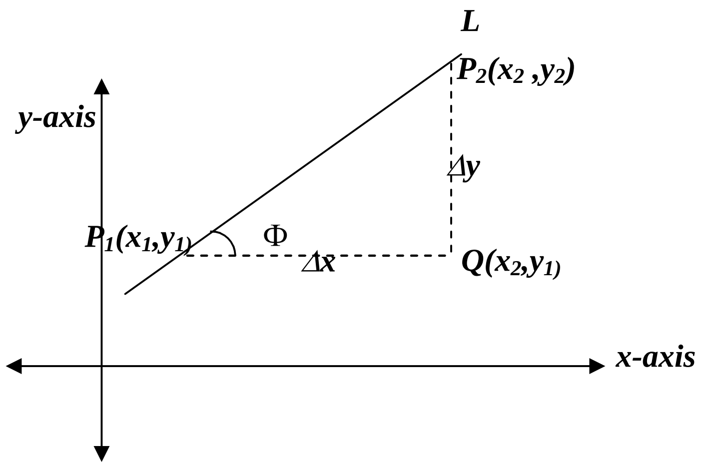
- A line that goes uphill as x increases has a positive slope . A line that goes downhill as x increases has a negative slope .
- A horizontal line has slope zero because $\Delta y = 0$.
- The slope of a vertical line is undefined because $\Delta x = 0$.
- Parallel lines have same slope .
- If neither of two perpendicular lines L_1 and L_2 is vertical , their slopes m_1 and m_2 are related by the equation : $m_1 . m_2 = -1$.

__Angles of Inclination__: The angle of inclination of a line that crosses the *x-axis* is the smallest angle we get when we measure counter clock from the *x-axis* around the point of intersection .

The slope of a line is the tangent of the line angle of inclination .

$m = \tan \Phi$ where Φ is the angle of inclination .

- The angle of inclination of a horizontal line is taken to be 0^o .
- Parallel lines have equal angle of inclination .



__*EX-1*__- Find the slope of the line determined by two points *A(2,1)* and *B(-1,3)* and find the common slope of the line perpendicular to *AB*.

__*Sol.*__- Slope of *AB* is: $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-1-2} = -\frac{2}{3}$

Slope of line perpendicular to *AB* is : $-\frac{1}{m_{AB}} = \frac{3}{2}$

__*EX-2*__- Use slopes to determine in each case whether the points are collinear (lie on a common straight line) :

a) *A(1,0) , B(0,1) , C(2,1) .*

b) *A(-3,-2) , B(-2,0) , C(-1,2) , D(1,6) .*

Sol. –

a) $m_{AB} = \frac{1-0}{0-1} = -1$ *and* $m_{BC} = \frac{1-1}{2-0} = 0 \neq m_{AB}$

The points *A* , *B* and *C* are not lie on a common straight line .

b) $m_{AB} = \frac{0-(-2)}{-2-(-3)} = 2$, $m_{BC} = \frac{2-0}{-1-(-2)} = 2$, $m_{CD} = \frac{6-2}{1-(-1)} = 2$

Since $m_{AB} = m_{BC} = m_{CD}$

Hence the points *A* , *B* , *C* , and *D* are collinear .

1-3- Equations for lines : An equation for a line is an equation that is satisfied by the coordinates of the points that lies on the line and is not satisfied by the coordinates of the points that lie elsewhere .

Vertical lines **: Every vertical line *L* has to cross the x-axis at some point *(a,0)*. The other points on *L* lie directly above or below *(a,0)* . This mean that :** $x = a \quad \forall (x, y)$

Nonvertical lines : **That point – slope equation of the line through the point** (x_1, y_1) **with slope *m* is :**

$$y - y_1 = m(x - x_1)$$

Horizontal lines : **The standard equation for the horizontal line through the point** (a, b) **is :** $y = b$.

The distance from a point to a line : **To calculate the distance *d* between the point** $P(x_1, y_1)$ **and** $Q(x_2, y_2)$ **is :**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We use this formula when the coordinate axes are scaled in a common unit .

To find the distance from the point $P(x_1, y_1)$ **to the line *L* , we follow :**

1. **Find an equation for the line *L'* through *P* perpendicular to *L* :**

   $y - y_1 = m'(x - x_1)$ **where** $m' = -1/m$

2. **Find the point** $Q(x_2, y_2)$ **by solving the equation for *L* and *L'* simultaneously .**
3. **Calculate the distance between *P* and *Q* .**

The general linear equation :

$Ax + By = C$ **where *A* and *B* not both zero.**

EX-3 **– Write an equation for the line that passes through point :**

a) $P(-1, 3)$ **with slope** $m = -2$ **.**

b) $P_1(-2, 0)$ **and** $P_2(2, -2)$**.**

Sol. **-** a) $y - y_1 = m(x - x_1) \rightarrow y - 3 = -2(x - (-1)) \rightarrow y + 2x = 1$

b)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-0}{2-(-2)} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1) \Rightarrow \quad y - 0 = -\frac{1}{2}(x - (-2)) \Rightarrow \quad 2y + x + 2 = 0$$

EX-4 **- Find the slope of the line :** $3x + 4y = 12$ **.**

Sol. **-** $y = -\frac{3}{4}x + 3 \Rightarrow \quad the \quad slope \quad is \quad m = -\frac{3}{4}$

EX-5 **- Find :**

a) an equation for the line through $P(2,1)$ **parallel to** $L: \; y = x + 2$ **.**

b) an equation for the line through P **perpendicular to** L **.**

c) the distance from P **to** L **.**

Sol. **-**

a)

$$since \quad L_2 // L_1 \Rightarrow \quad m_{L2} = m_{L1} = 1 \Rightarrow \quad y - 1 = 1(x - 2) \quad \Rightarrow \quad y = x - 1$$

b) Since L_1 **and** L_3 **are perpendicular lines then :**

$$m_{L3} = -1 \Rightarrow \quad y - 1 = -(x - 2) \Rightarrow \quad y + x = 3$$

c)

$$\begin{matrix} y = x + 2 \\ y + x = 3 \end{matrix} \quad \Rightarrow \quad x = \frac{1}{2} \quad and \quad y = \frac{5}{2} \quad \Rightarrow \quad P(2,1) \quad and \quad Q\left(\frac{1}{2},\frac{5}{2}\right)$$

$$\Rightarrow d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} = \sqrt{4.5}$$

EX-6 **– Find the angle of inclination of the line :** $\sqrt{3}x + y = -3$

Sol. **-**

$$y = -\sqrt{3}x - 3 \quad \Rightarrow \quad m = -\sqrt{3}$$

$$m = tan\Phi = -\sqrt{3} \quad \Rightarrow \quad \Phi = 120^o$$

EX-7 **- Find the line through the point** $P(1, 4)$ **with the angle of inclination** $\Phi = 60^o$ **.**

Sol. **-**

$$m = tan\Phi = tan60 = \sqrt{3}$$

$$y - 4 = \sqrt{3}(x - 1) \Rightarrow \quad y = \sqrt{3}x + 4 - \sqrt{3}$$

EX-8 **- The pressure** P **experienced by a diver under water is related to the diver's depth** d **by an equation of the form** $P = k\,d + 1$ **where** k **a constant . When** $d = 0$ **meters , the pressure is** 1 **atmosphere . The pressure at** 100 **meters is about** 10.94 **atmosphere . Find the pressure at** 50 **meters.**

Sol. **- At** $P = 10.94$ **and** $d = 100 \quad \rightarrow \quad 10.94 = k(100)+1 \rightarrow k = 0.0994$

$P = 0.0994\,d + 1$ **, at** $d = 50 \quad \rightarrow \quad P = 0.0994 * 50 + 1 = 5.97$ **atmo.**

1-4- <u>Functions</u> : ***Function*** **is any rule that assigns to each element in one set some element from another set :**

$$y = f(x)$$

The inputs make up the *domain of the function* , and the outputs make up *the function's range.*

The variable x is called *independent variable of the function* , and the variable y whose value depends on x is called *the dependent variable of the function* .

We must keep two restrictions in mind when we define functions :

1. **We never divide by zero .**
2. **We will deal with real – valued functions only.**

<u>Intervals</u> :

- **The *open interval* is the set of all real numbers that be strictly between two fixed numbers a and b :**

$$(a,b) \equiv a < x < b$$

- **The *closed interval* is the set of all real numbers that contain both endpoints :**

$$[a,b] \equiv a \le x \le b$$

- ***Half open interval* is the set of all real numbers that contain one endpoint but not both :**

$$[a,b) \equiv a \le x < b$$

$$(a,b] \equiv a < x \le b$$

<u>Composition of functions</u> : **suppose that the outputs of a function f can be used as inputs of a function g . We can then hook f and g together to form a new function whose inputs are the inputs of f and whose outputs are the numbers :**

$$(g_o f)(x) = g(f(x))$$

***<u>EX-9</u>*- Find the domain and range of each function :**

$a)\quad y = \sqrt{x+4} \quad , \quad b)\quad y = \dfrac{1}{x-2}$

$c)\quad y = \sqrt{9-x^2} \quad , \quad d)\quad y = \sqrt{2-\sqrt{x}}$

<u>Sol.</u> - $a)\quad x+4 \ge 0 \Rightarrow \quad x \ge -4 \Rightarrow \quad D_x : \forall x \ge -4 \quad , \quad R_y : \forall y \ge 0$

$b)\quad x-2 \ne 0 \Rightarrow \quad x \ne 2 \Rightarrow \quad D_x : \forall x \ne 2$

$y = \dfrac{1}{x-2} \Rightarrow \quad x = \dfrac{1}{y} + 2 \Rightarrow \quad R_y : \forall y \ne 0$

$c)\quad 9 - x^2 \ge 0 \Rightarrow \quad -3 \le x \le 3 \Rightarrow \quad D_x : -3 \le x \le 3$

$y = \sqrt{9-x^2} \Rightarrow \quad x = \mp\sqrt{9-y^2}$

$since \quad 9 - y^2 \ge 0 \Rightarrow \quad -3 \le y \le 3$

$since \quad y \ge 0 \quad \Rightarrow \quad R_y : 0 \le y \le 3$

$$d)\quad 2-\sqrt{x}\ge 0 \Rightarrow \quad 0\le x\le 4 \Rightarrow \quad D_x: 0\le x\le 4$$

$$\begin{matrix} if & x=0\Rightarrow & y=\sqrt{2} \\ if & x=4\Rightarrow & y=0 \end{matrix} \quad \Rightarrow \quad R_y: 0\le y\le \sqrt{2}$$

EX-10- Let $f(x)=\dfrac{x}{x-1}$ and $g(x)=1+\dfrac{1}{x}$.

Find $(g_o f)(x)$ and $(f_o g)(x)$.

Sol.-

$$(g_o f)(x)=g(f(x))=g\left(\frac{x}{x-1}\right)=1+\frac{1}{\frac{x}{x-1}}=\frac{2x-1}{x}$$

$$(f_o g)(x)=f(g(x))=f\left(1+\frac{1}{x}\right)=\frac{1+\frac{1}{x}}{1+\frac{1}{x}-1}=x+1$$

EX-11- Let $(g_o f)(x)=x$ and $f(x)=\dfrac{1}{x}$. Find $g(x)$.

Sol.- $(g_o f)(x)=g\left(\dfrac{1}{x}\right)=x\Rightarrow \quad g(x)=\dfrac{1}{x}$

1-5- Limits and continuity :

Limits **:** The limit of $F(t)$ as t approaches C is the number L if : Given any radius $\varepsilon>0$ about L there exists a radius $\delta>0$ about C such that for all t, $0<|t-C|<\delta$ implies $|F(t)-L|<\varepsilon$ and we can write it as :

$$\lim_{t\to C} F(t)=L$$

The limit of a function $F(t)$ as $t\to C$ never depend on what happens when $t=C$.

Right hand limit : $\lim\limits_{t\to C^+} F(t)=L$

The limit of the function $F(t)$ as $t\to C$ from the right equals L if : Given any $\varepsilon>0$ (radius about L) there exists a $\delta>0$ (radius to the right of C) such that for all t :

$$C<t<C+\delta \Rightarrow \quad |F(t)-L|<\varepsilon$$

Left hand limit : $\lim\limits_{t\to C^-} F(t)=L$

The limit of the function $F(t)$ as $t\to C$ from the left equal L if : Given any $\varepsilon>0$ there exists a $\delta>0$ such that for all t :

$$C-\delta<t<C \Rightarrow \quad |F(t)-L|<\varepsilon$$

Note that – **A function $F(t)$ has a limit at point C if and only if the right hand and the left hand limits at C exist and equal . In symbols :**

$$\lim_{t\to C} F(t) = L \quad \Leftrightarrow \quad \lim_{t\to C^+} F(t) = L \quad \text{and} \quad \lim_{t\to C^-} F(t) = L$$

The limit combinations theorems :

$$1)\quad \lim[F_1(t) \mp F_2(t)] = \lim F_1(t) \mp \lim F_2(t)$$

$$2)\quad \lim[F_1(t) * F_2(t)] = \lim F_1(t) * \lim F_2(t)$$

$$3)\quad \lim \frac{F_1(t)}{F_2(t)} = \frac{\lim F_1(t)}{\lim F_2(t)} \qquad \text{where} \quad \lim F_2(t) \neq 0$$

$$4)\quad \lim[k * F_1(t)] = k * \lim F_1(t) \qquad \forall k$$

$$5)\quad \lim_{\theta\to 0} \frac{Sin\theta}{\theta} = 1$$

provided that θ is measured in radius

The limits (in *1 – 4*) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

Thm. -1 : The sandwich theorem **: Suppose that $f(t) \leq g(t) \leq h(t)$ for all $t \neq C$ in some interval about C and that $f(t)$ and $h(t)$ approaches the same limit L as $t \to C$, then :**

$$\lim_{t\to C} g(t) = L$$

Infinity as a limit :

1.The limit of the function $f(x)$ as x approaches infinity is the number L: $\lim_{x\to\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number M such that for all x : $M < x \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

2. The limit of $f(x)$ as x approaches negative infinity is the number L : $\lim_{x\to -\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number N such that for all x : $x < N \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

The following facts are some times abbreviated by saying :

- **a) As x approaches 0 from the right , $1/x$ tends to ∞ .**
- **b) As x approaches 0 from the left , $1/x$ tends to $-\infty$.**
- **c) As x tends to ∞ , $1/x$ approaches 0 .**
- **d) As x tends to $-\infty$, $1/x$ approaches 0 .**

Continuity :

Continuity at an interior point **: A function $y = f(x)$ is continuous at an interior point C of its domain if : $\lim_{x\to C} f(x) = f(C)$.**

Continuity at an endpoint **: A function $y = f(x)$ is continuous at a left endpoint a of its domain if : $\lim_{x\to a^+} f(x) = f(a)$.**

A function $y = f(x)$ is continuous at a right endpoint b of its domain if: $\lim_{t\to b^-} f(x) = f(b)$.

Continuous function : **A function is continuous if it is continuous at each point of its domain .**

Discontinuity at a point : **If a function f is not continuous at a point C, we say that f is discontinuous at C, and call C a point of discontinuity of f .**

The continuity test : **The function $y = f(x)$ is continuous at $x = C$ if and only if all three of the following statements are true :**

1) $f(C)$ exist (C is in the domain of f) .
2) $\lim_{x\to C} f(x)$ exists (f has a limit as $x\to C$) .
3) $\lim_{x\to C} f(x) = f(C)$ (the limit equals the function value) .

Thm.-2 : The limit combination theorem for continuous function :

If the function f and g are continuous at $x = C$, then all of the following combinations are continuous at $x = C$:

$1)\ f \mp g \quad 2)\ f.g \quad 3)\ k.g\ \ \forall k \quad 4)\ g_o f\ ,\ f_o g \quad 5)\ f/g$

provided $g(C) \neq 0$

Thm.-3 : **A function is continuous at every point at which it has a derivative . That is , if $y = f(x)$ has a derivative $f'(C)$ at $x = C$, then f is continuous at $x = C$.**

EX-12 **– Find :**

$$1)\ \lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}\quad ,\quad 2)\ \lim_{x\to a}\frac{x^3-a^3}{x^4-a^4}$$

$$3)\ \lim_{x\to 0}\frac{Sin5x}{Sin3x}\quad ,\quad 4)\ \lim_{y\to 0}\frac{tan2y}{3y}$$

$$5)\ \lim_{x\to 0}\frac{Sin2x}{2x^2+x}\quad ,\quad 6)\ \lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)$$

$$7)\ \lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5}\quad ,\quad 8)\ \lim_{y\to\infty}\frac{3y+7}{y^2-2}$$

$$9)\ \lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5}\quad ,\quad 10)\ \lim_{x\to -1^-}\frac{1}{x+1}$$

$$11)\ \lim_{x\to 0}Cos\left(1-\frac{Sinx}{x}\right)\quad ,\quad 12)\ \lim_{x\to 0}Sin\left(\frac{\pi}{2}Cos(tan x)\right)$$

S0l.-

$$1)\ \lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}=\lim_{x\to 0}\frac{5x+8}{3x^2-16}=\frac{0+8}{0-16}=-\frac{1}{2}$$

$$2)\ \lim_{x\to a}\frac{x^3-a^3}{x^4-a^4}=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)(x^2+a^2)}=\frac{a^2+a^2+a^2}{(a+a)(a^2+a^2)}=\frac{3}{4a}$$

$$3)\ \lim_{x\to 0}\frac{5\dfrac{Sin5x}{5x}}{3\dfrac{Sin3x}{3x}}=\frac{5}{3}.\frac{\lim\limits_{5x\to 0}\dfrac{Sin5x}{5x}}{\lim\limits_{3x\to 0}\dfrac{Sin3x}{3x}}=\frac{5}{3}$$

4) $\lim_{y\to 0}\frac{\tan 2y}{3y}=\frac{2}{3}.\lim_{2y\to 0}\frac{Sin2y}{2y}.\lim_{y\to 0}\frac{1}{Cos2y}=\frac{2}{3}$

5) $\lim_{x\to 0}\frac{Sin2x}{2x^2+x}=2\lim_{2x\to 0}\frac{Sin2x}{2x}.\lim_{x\to 0}\frac{1}{2x+1}=2$

6) $\lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)=1+Cos0=2$

7) $\lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5}=\lim_{x\to\infty}\frac{3+\frac{5}{x}-\frac{7}{x^3}}{10-\frac{11}{x^2}+\frac{5}{x^3}}=\frac{3}{10}$

8) $\lim_{y\to\infty}\frac{3y+7}{y^2-2}=\lim_{y\to\infty}\frac{\frac{3}{y}+\frac{7}{y^2}}{1-\frac{2}{y^2}}=\frac{0}{1}=0$

9) $\lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5}=\lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{\frac{2}{x}-\frac{7}{x^2}+\frac{5}{x^3}}=\frac{1}{0}=\infty$

10) $\lim_{x\to -1^-}\frac{1}{x+1}=\frac{1}{-1+1}=-\infty$

11) $\lim_{x\to 0}Cos\left(1-\frac{Sinx}{x}\right)=Cos\left(1-\lim_{x\to 0}\frac{Sinx}{x}\right)=Cos0=1$

12) $\lim_{x\to 0}Sin\left(\frac{\pi}{2}Cos(\tan x)\right)=Sin\left(\frac{\pi}{2}Cos(\tan 0)\right)=Sin\left(\frac{\pi}{2}Cos0\right)=Sin\frac{\pi}{2}=1$

EX-13- Test continuity for the following function :

$$f(x)=\begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0\le x<1 \\ 1 & x=1 \\ -2x+4 & 1<x\le 2 \\ 0 & 2<x\le 3 \end{cases}$$

Sol.- We test the continuity at midpoints $x = 0 , 1 , 2$ and endpoints $x = -1 , 3$.

$At \quad x=0 \Rightarrow \qquad f(0)=2*0=0$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0}(x^2-1)=-1$$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0}2x=0\neq\lim_{x\to 0^-}f(x)$$

$$Since \quad \lim_{x\to 0}f(x) \ doesn't\ exist$$

$$Hence \quad the\ function\ discontinuous\ at\ x=0$$

$At \quad x=1 \Rightarrow \quad f(1)=1$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1} 2x=2$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(-2x+4)=2=\lim_{x\to 1^-} f(x)=\lim_{x\to 1} f(x)$$

$$Since \quad \lim_{x\to 1} f(x)\neq f(1)$$

Hence the function is discontinuous at $x=1$

$At \quad x=2 \Rightarrow \quad f(2)=-2*2+4=0$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(-2x+4)=0$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2} 0=0=\lim_{x\to 2^-} f(x)=\lim_{x\to 2} f(x)$$

$$Since \quad \lim_{x\to 2} f(x)=f(2)=0$$

Hence the function is continuous at $x=2$

$At \quad x=-1 \Rightarrow \quad f(-1)=(-1)^2-1=0$

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1}(x^2-1)=0=f(-1)$$

Hence the function is continuous at $x=-1$

$At \quad x=3 \Rightarrow \quad f(3)=0$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 0=0=f(3)$$

Hence the function is continuous at $x=3$

<u>*EX-14*</u>- What value should be assigned to *a* to make the function :

$$f(x)=\begin{cases} x^2-1 & x<3 \\ 2ax & x\geq 3 \end{cases}$$ **continuous at $x = 3$?**

<u>Sol.</u> –

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)\Rightarrow \lim_{x\to 3}(x^2-1)=\lim_{x\to 3} 2ax \Rightarrow \quad 8=6a \Rightarrow \quad a=\frac{4}{3}$$

Problems – 1

1. The steel in railroad track expands when heated . For the track temperature encountered in normal outdoor use , the length S of a piece of track is related to its temperature t by a linear equation . An experiment with a piece of track gave the following measurements :

$$t_1 = 65^o F \quad , \quad S_1 = 35 \ ft$$
$$t_2 = 135^o F \quad , \quad S_2 = 35.16 \ ft$$

Write a linear equation for the relation between S and t .

(ans.: S=0.0023t+34.85)

2. Three of the following four points lie on a circle center the origin . Which are they , and what is the radius of the circle ?
A(-1.7) , B(5,-5) , C(-7,5) and D(7,-1). *(ans.: A,B,D;√50)*

3. *A* and *B* are the points *(3,4)* and *(7,1)* respectively . Use Pythagoras theorem to prove that *OA* is perpendicular to *AB* . Calculate the slopes of *OA* and *AB* , and find their product . *(ans.: 4/3, -3/4;-1)*

4. *P(-2,-4)* , *Q(-5,-2)* , *R(2,1)* and *S* are the vertices of a parallelogram . Find the coordinates of *M* , the point of intersection of the diagonals and of *S.*
(ans.: M(0,-3/2) , S(5,-1))

5. Calculate the area of the triangle formed by the line $3x-7y+4 =0$, and the axes . *(ans.: 8/21)*

6. Find the equation of the straight line through *P(7,5)* perpendicular to the straight line *AB* whose equation is $3x + 4y -16 = 0$. Calculate the length of the perpendicular from *P* and *AB.* *(ans.: 3y-4x+13=0;5)*

7. *L(-1,0)* , *M(3,7)* and *N(5,-2)* are the mid-points of the sides *BC* , *CA* and *AB* respectively of the triangle *ABC.* Find the equation of *AB.* *(ans.:4y=7x-43)*

8. The straight line $x – y – 6 = 0$ cuts the curve $y^2 = 8x$ at *P* and *Q* . Calculate the length of *PQ* . *(ans.:16√2)*

9. A line is drawn through the point *(2,3)* making an angle of 45^o with the positive direction of the x-axis and it meets the line $x = 6$ at *P* . Find the distance of *P* from the origin *O* , and the equation of the line through *P* perpendicular to *OP.* *(ans.: √85,7y+6x-85=0)*

10. The vertices of a quadrilateral *ABCD* are *A(4,0)* , *B(14,11)* , *C(0,6)* and *D(-10,-5)* . Prove that the diagonals *AC* and *BD* bisect each other at right angles , and that the length of *BD* is four times that of *AC* .

11. The coordinates of the vertices *A, B* and *C* of the triangle *ABC* are *(-3,7)*, *(2,19)* and *(10,7)* respectively :

a) Prove that the triangle is isosceles.

b) Calculate the length of the perpendicular from *B* to *AC* , and use it to find the area of the triangle . ***(ans.:12,78)***

12. Find the equations of the lines which pass through the point of intersection of the lines $x - 3y = 4$ and $3x + y = 2$ and are respectively parallel and perpendicular to the line $3x + 4y = 0$.

(ans.:4y+3x+1=0;3y-4x+7=0)

13. Through the point *A(1,5)* is drawn a line parallel to the x-axis to meet at *B* the line *PQ* whose equation is $3y = 2x - 5$. Find the length of *AB* and the sine of the angle between *PQ* and *AB* ; hence show that the length of the perpendicular from *A* to *PQ* is $18/\sqrt{13}$. Calculate the area of the triangle formed by *PQ* and the axes . ***(ans.:9,2/√13,25/12)***

14. Let $y = \dfrac{x^2 + 2}{x^2 - 1}$, express x in terms of y and find the values of y for which x is real . **(ans.: $x = \mp\sqrt{\dfrac{y+2}{y-1}}; y \le -2 \;\; or \;\; y > 1$)**

15. Find the domain and range of each function :

$$a)\; y = \frac{1}{1 + x^2} \quad , \quad b)\; y = \frac{1}{1 + \sqrt{x}} \quad , \quad c)\; y = \frac{1}{\sqrt{3 - x}}$$

$(ans.: a)\forall x, 0 < y \le 1;\; b) x \ge 0, y > 0;\; c) x < 3, y > 0)$

16. Find the points of intersection of $x^2 = 4y$ and $y = 4x$. ***(ans.:(0,0),(16,64))***

17. Find the coordinates of the points at which the curves cut the axes :

$$a)\; y = x^3 - 9x^2 \quad , \quad b)\; y = (x^2 - 1)(x^2 - 9) \quad , \quad c)\; y = (x + 1)(x - 5)^2$$

(ans.:a)(0,0);(0,0),(9,0);b)(0,9);(1,0),(-1,0),(3,0),(-3,0);c)(0,25);(-1,0),(5,0))

18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants *a* , *b* , *c* and *d* to make *f(g(x))* and *g(f(x))* identical ?

(ans.:ad+b=bc+d)

19. A particle moves in the plane from *(-2,5)* to the y-axis in such away that $\Delta y = 3 * \Delta x$. Find its new coordinates . ***(ans.:(0,11),(0,-1))***

20. If $f(x) = 1/x$ and $g(x) = 1/\sqrt{x}$, what are the domain of *f , g , f+g , f-g , f.g , f/g , g/f* , $f_o g$ and $g_o f$? What is the domain of $h(x) = g(x+4)$?

$(ans.: \forall x \ne 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x \ge 0, \forall x \ge 0, \forall x \ge 0; \forall x > -4)$

21. Discuss the continuity of :

$$f(x)=\begin{cases} x+\dfrac{1}{x} & \text{for} \quad x<0 \\ -x^3 & \text{for} \quad 0\le x<1 \\ -1 & \text{for} \quad 1\le x<2 \\ 1 & \text{for} \quad x=2 \\ 0 & \text{for} \quad x>2 \end{cases}$$

(ans.: discontinuous at x=0,2 ; continuous at x=1)

22. Evaluate the following limits :

a) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{2x+5}$ b) $\lim\limits_{x\to\infty}\dfrac{1+\sin x}{x}$

c) $\lim\limits_{x\to 0}\dfrac{x}{\tan 3x}$ d) $\lim\limits_{x\to\infty}\dfrac{x.\sin x}{(x+\sin x)^2}$

e) $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{1-x}$ f) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

g) $\lim\limits_{n\to\infty}(\sqrt{n^2+1}-n)$

(ans.:a)1/2, b)0, c)1/3, d)0, e)1/2, f)-1/2√2, g)0)

23. Suppose that : $f(x)=x^3-3x^2-4x+12$ **and** $h(x)=\begin{cases} \dfrac{f(x)}{x-3} & \text{for } x\ne 3 \\ k & \text{for } x=3 \end{cases}$.

Find : a) all zeros of f **.**

b) the value of k **that makes** h **continuous at** $x=3$ **.**

$(ans.: a)\,x=\mp 2,3;\ b)\,k=5)$

Chapter two
Functions

2-1- <u>Exponential and Logarithm functions</u> :

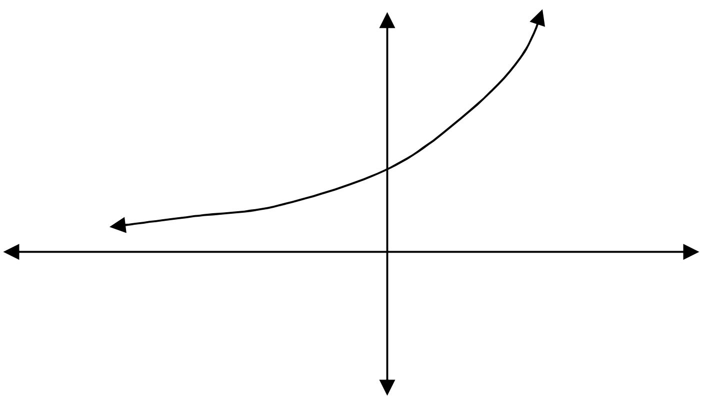
Exponential functions : **If a is a positive number and x is any number , we define the exponential function as :**

$$y = a^x \quad \text{with domain : } -\infty < x < \infty$$
$$\text{Range : } y > 0$$

The properties of the exponential functions are :

1. If $a > 0 \leftrightarrow a^x > 0$.
2. $a^x . a^y = a^{x+y}$.
3. $a^x / a^y = a^{x-y}$.
4. $(a^x)^y = a^{x.y}$.
5. $(a . b)^x = a^x . b^x$.
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$.
7. $a^{-x} = 1 / a^x$ and $a^x = 1 / a^{-x}$.
8. $a^x = a^y \leftrightarrow x = y$.
9. $a^0 = 1$,
   $a^\infty = \infty$, $a^{-\infty} = 0$, where $a > 1$.
   $a^\infty = 0$, $a^{-\infty} = \infty$, where $a < 1$.

The graph of the exponential function $y = a^x$ is :



Logarithm function : **If a is any positive number other than 1 , then the logarithm of x to the base a denoted by :**

$$y = \log_a x \quad \text{where } x > 0$$

At $a = e = 2.7182828\ldots$, we get the natural logarithm and denoted by :

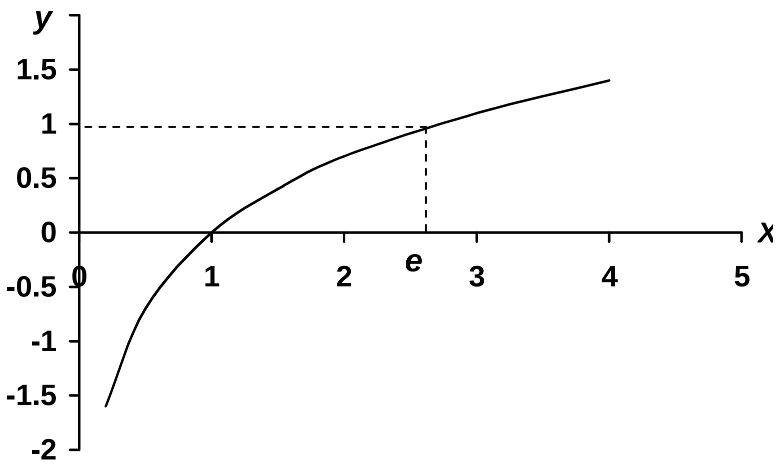
$$y = \ln x$$

Let $x , y > 0$ then the properties of logarithm functions are :

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.

4. $\ln (x.y) = \ln x + \ln y$.
5. $\ln (x / y) = \ln x - \ln y$.
6. $\ln x^n = n. \ln x$.
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x. \ln a}$.
9. $e^{\ln x} = x$.

The graph of the function $y = \ln x$ is :



Application of exponential and logarithm functions **:**

We take Newton's law of cooling :

$$T - T_S = (T_0 - T_S) e^{t k}$$

where T is the temperature of the object at time t .
T_S is the surrounding temperature .
T_0 is the initial temperature of the object .
k is a constant .

***EX-1*- The temperature of an ingot of metal is $80\ ^oC$ and the room temperature is $20\ ^oC$. After twenty minutes, it was $70\ ^oC$.**

a) What is the temperature will the metal be after *30* minutes?
b) What is the temperature will the metal be after two hours?
c) When will the metal be $30\ ^oC$?

Sol. **:**

$$T - T_S = (T_0 - T_S)e^{tk} \Rightarrow 50 = 60e^{20k} \Rightarrow k = \frac{\ln 5 - \ln 6}{20} = -0.0091$$

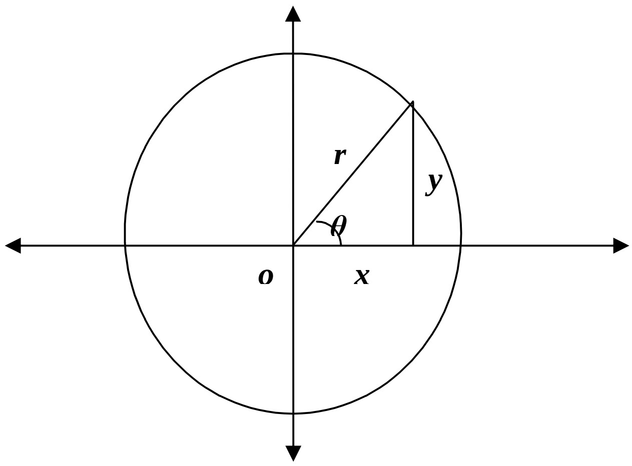
$$a) \quad T - 20 = 60e^{30(-0.0091)} = 60 * 0.761 = 45.6\ ^oC \Rightarrow T = 65.6\ ^oC$$

$$b) \quad T - T_S = 60e^{120(-0.0091)} = 60 * 0.335 = 20.1\ ^oC \Rightarrow T = 40.1\ ^oC$$

$$c) \quad 10 = 60e^{-0.0091\, t} \Rightarrow -0.0091\, t = -\ln 6 \Rightarrow t = 3.3\ hrs.$$

2-2- <u>Trigonometric functions</u> : When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :

$$Sin\theta = \frac{y}{r} = \frac{1}{csc\,\theta} \quad , \quad Cos\theta = \frac{x}{r} = \frac{1}{sec\,\theta} \quad , \quad tan\,\theta = \frac{y}{x} = \frac{1}{Cot\theta} = \frac{Sin\theta}{Cos\theta}$$



The following are some properties of these functions :

1) $Sin^2\theta + Cos^2\theta = 1$

2) $1 + tan^2\,\theta = sec^2\,\theta$ and $1 + Cot^2\theta = csc^2\,\theta$

3) $Sin(\theta \mp \beta) = Sin\theta.Cos\beta \mp Cos\theta.Sin\beta$

4) $Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\theta.Sin\beta$

5) $tan(\theta \mp \beta) = \dfrac{tan\,\theta \mp tan\,\beta}{1 \pm tan\,\theta.tan\,\beta}$

6) $Sin2\theta = 2Sin\theta.Cos\theta$ and $Cos2\theta = Cos^2\theta - Sin^2\theta$

7) $Cos^2\theta = \dfrac{1 + Cos2\theta}{2}$ and $Sin^2\theta = \dfrac{1 - Cos2\theta}{2}$

8) $Sin(\theta \mp \frac{\pi}{2}) = \mp Cos\theta$ and $Cos(\theta \mp \frac{\pi}{2}) = \pm Sin\theta$

9) $Sin(-\theta) = -Sin\theta$ and $Cos(-\theta) = Cos\theta$ and $tan(-\theta) = -tan\,\theta$

10) $Sin\theta.Sin\beta = \frac{1}{2}[Cos(\theta - \beta) - Cos(\theta + \beta)]$

$Cos\theta.Cos\beta = \frac{1}{2}[Cos(\theta - \beta) + Cos(\theta + \beta)]$

$Sin\theta.Cos\beta = \frac{1}{2}[Sin(\theta - \beta) + Sin(\theta + \beta)]$

$$11)\quad Sin\theta + Sin\beta = 2Sin\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

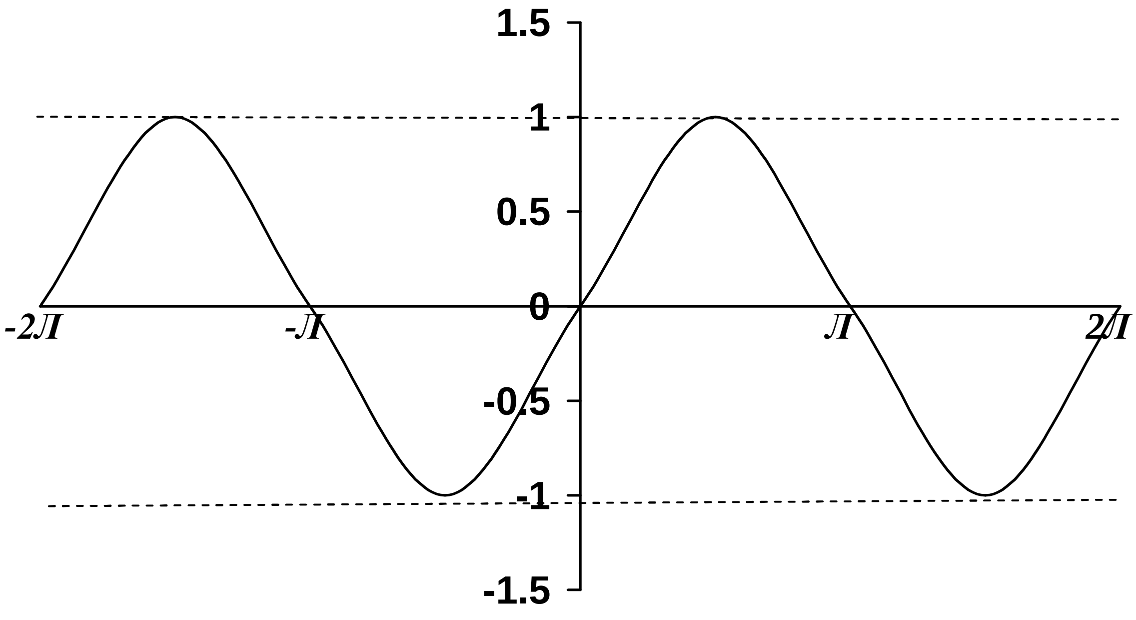
$$Sin\theta - Sin\beta = 2Cos\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

$$12)\quad Cos\theta + Cos\beta = 2Cos\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

$$Cos\theta - Cos\beta = -2Sin\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

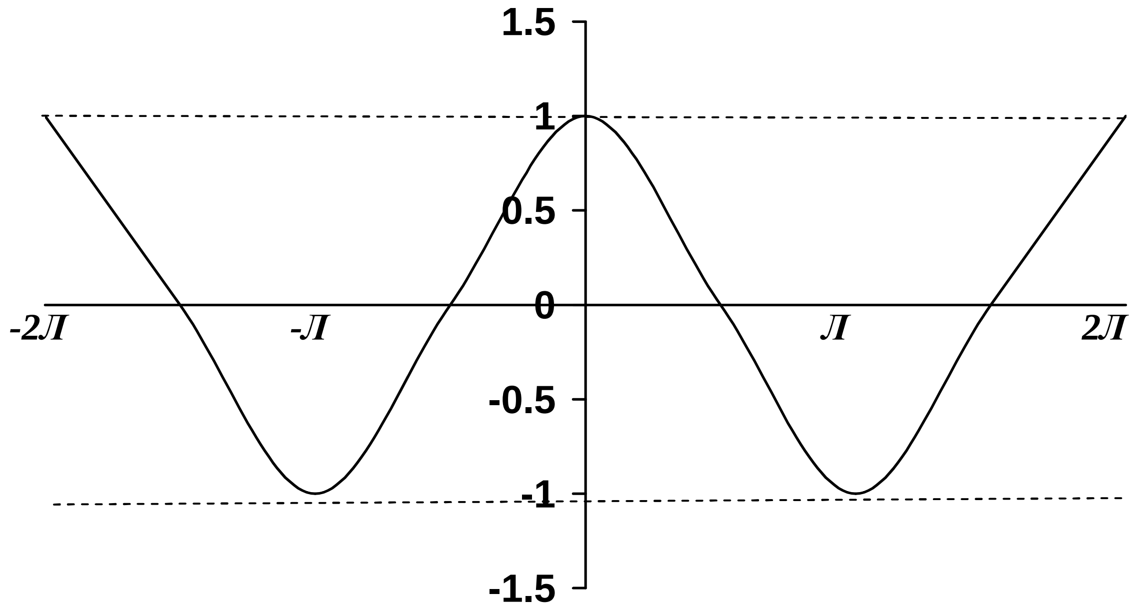
θ	*0*	*Π / 6*	*Π / 4*	*Π / 3*	*Π / 2*	*Π*
Sinθ	*0*	*1/2*	*1/√2*	*√3/2*	*1*	*0*
Cosθ	*1*	*√3/2*	*1/√2*	*1/2*	*0*	*-1*
tanθ	*0*	*1/√2*	*1*	*√3*	∞	*0*

Graphs of the trigonometric functions are :

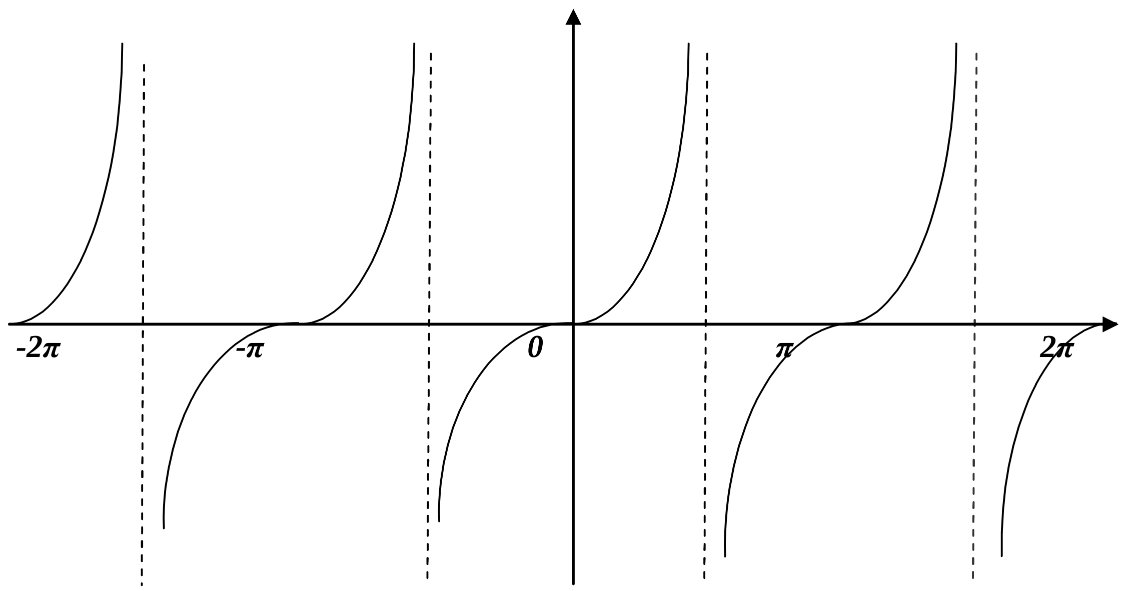


$$y = Sinx \quad D_x : \forall x$$

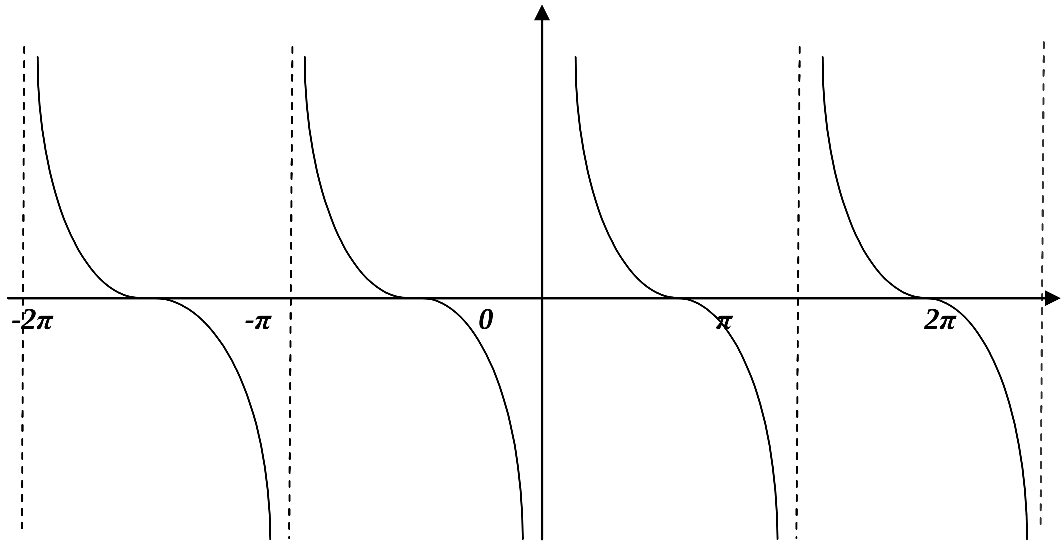
$$R_y : -1 \le y \le 1$$



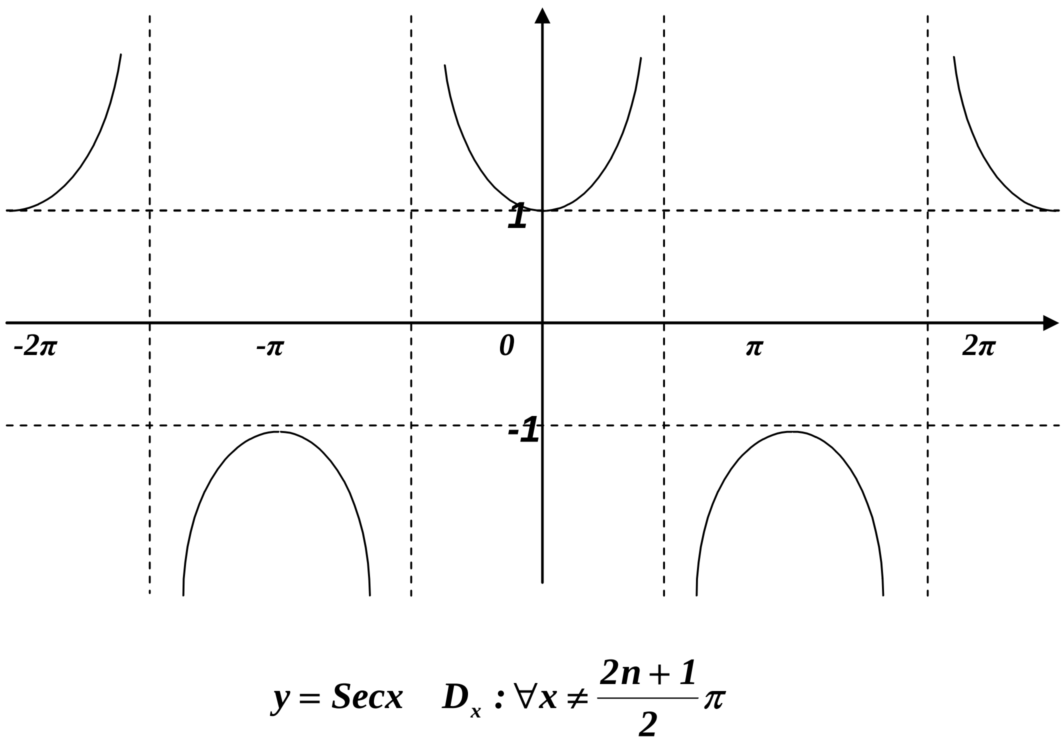
$$y = Cosx \quad D_x : \forall x$$
$$R_y : -1 \le y \le 1$$



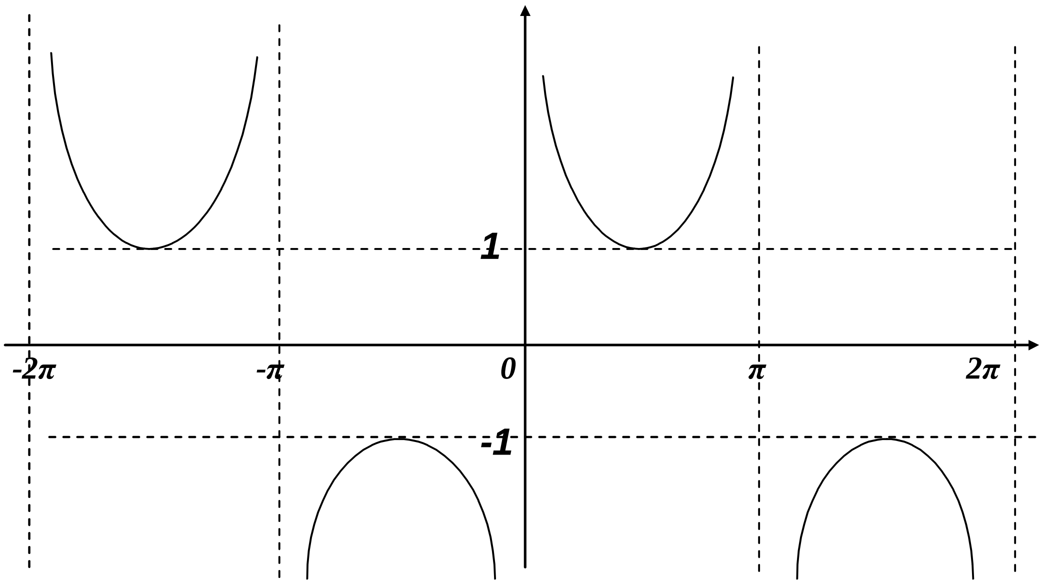
$$y = \tan x \quad D_x : \forall x \ne \frac{2n+1}{2}\pi$$
$$R_y : \forall y$$



$$y = Cotx \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y$$



$$y = Secx \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y \geq 1 \;\; or \;\; y \leq -1$$



$y = csc\,x \quad D_x : \forall x \neq n\pi$

$R_y : \forall y \geq 1 \;\; or \;\; y \leq -1$

$Where \;\; n = 0, \mp 1, \mp 2, \mp 3, ...$

<u>*EX-2*</u> - Solve the following equations , for values of *θ* from 0^o to 360^o inclusive .

a) $tan\ \theta = 2\ Sin\ \theta$ **b)** $1 + Cos\ \theta = 2\ Sin^2\ \theta$

<u>*Sol.*</u>-

$$a) \quad tan\theta = 2Sin\theta \Rightarrow \frac{Sin\theta}{Cos\theta} = 2Sin\theta$$

$$\Rightarrow Sin\theta(1 - 2Cos\theta) = 0$$

$$either \;\; Sin\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o$$

$$or \quad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

Therefore the required values of *θ* are $0^o, 60^o, 180^o, 300^o, 360^o$.

$$b) \quad 1 + Cos\theta = 2.Sin^2\theta \Rightarrow 1 + Cos\theta = 2(1 - Cos^2\theta)$$

$$\Rightarrow (2Cos\theta - 1)(Cos\theta + 1) = 0$$

$$either \qquad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

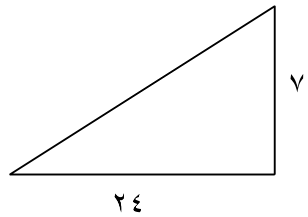
$$or \qquad Cos\theta = -1 \Rightarrow \theta = 180^o$$

There the roots of the equation between 0^o and 360^o are $60^o, 180^o$ and 300^o .

EX-3- **If** ***tan θ = 7/24,*** **find without using tables the values of** ***Secθ*** **and** ***Sinθ.***
Sol.-

$$\tan\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$Sec\theta = \frac{r}{x} = \frac{25}{24} \qquad and \qquad Sin\theta = \frac{y}{r} = \frac{7}{25}$$



EX-4- **Prove the following identities :**

$$a)\quad Csc\theta + \tan\theta.Sec\theta = Csc\theta.Sec^2\theta$$

$$b)\quad Cos^4\theta - Sin^4\theta = Cos^2\theta - Sin^2\theta$$

$$c)\quad \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta}$$

Sol.-

$$a)\quad L.H.S. = Csc\theta + \tan\theta.Sec\theta = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos\theta}.\frac{1}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Sin\theta.Cos^2\theta} = \frac{1}{Sin\theta}.\frac{1}{Cos^2\theta} = Csc\theta.Sec^2\theta = R.H.S.$$

$$b)\quad L.H.S. = Cos^4\theta - Sin^4\theta = (Cos^2\theta - Sin^2\theta).(Cos^2\theta + Sin^2\theta)$$

$$= Cos^2\theta - Sin^2\theta = R.H.S.$$

$$c)\quad L.H.S. = \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\frac{1}{Cos\theta} - \frac{1}{Sin\theta}}{\frac{Sin\theta}{Cos\theta} - \frac{Cos\theta}{Sin\theta}} = \frac{1}{Sin\theta + Cos\theta}$$

$$= \frac{Sin^2\theta + Cos^2\theta}{Sin\theta + Cos\theta}.\frac{\frac{1}{Sin\theta.Cos\theta}}{\frac{1}{Sin\theta.Cos\theta}} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta} = R.H.S.$$

EX-5- **Simplify** $\frac{1}{\sqrt{x^2 - a^2}}$ *when* $x = a.Csc\theta$.

Sol.- $\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{a^2Csc^2\theta - a^2}} = \frac{1}{a\sqrt{Cot^2\theta}} = \frac{1}{a}\tan\theta$.

EX-6- **Eliminate** ***θ*** **from the equations :**

i) $x = a\,Sin\theta$ *and* $y = b\tan\theta$

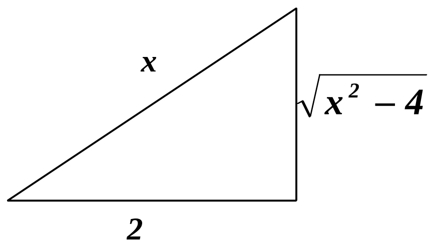
ii) $x = 2\,Sec\theta$ *and* $y = Cos2\theta$

Sol.-

$$i)\quad x = a.Sin\theta \Rightarrow Sin\theta = \frac{x}{a} \Rightarrow Csc\theta = \frac{a}{x}$$

$$y = b\tan\theta \Rightarrow \tan\theta = \frac{y}{b} \Rightarrow Cot\theta = \frac{b}{y}$$

$$Since \quad Csc^2\theta = Cot^2\theta + 1 \Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$$



$$ii)\quad x = 2\,Sec\,\theta \Rightarrow Cos\,\theta = \frac{2}{x}$$

$$y = Cos\,2\theta \Rightarrow y = Cos^2\theta - Sin^2\theta$$

$$y = \frac{4}{x^2} - \frac{x^2 - 4}{x^2} \Rightarrow x^2 y = 8 - x^2$$

EX-7- If $tan^2\theta - 2\ tan^2\beta = 1$, show that $2\ Cos^2\theta - Cos^2\beta = 0$.

Sol. –

$$tan^2\ \theta - 2\,tan^2\ \beta = 1 \Rightarrow Sec^2\theta - 1 - 2(Sec^2\beta - 1) = 1$$

$$\Rightarrow Sec^2\theta - 2Sec^2\beta = 0 \Rightarrow \frac{1}{Cos^2\theta} - \frac{2}{Cos^2\beta} = 0$$

$$\Rightarrow 2Cos^2\theta - Cos^2\beta = 0 \qquad Q.E.D.$$

EX-8- If $a\ Sin\theta = p - b\ Cos\theta$ and $b\ Sin\theta = q + a\ Cos\theta$ .Show that : $a^2 + b^2 = p^2 + q^2$

Sol.-

$$p = a.Sin\theta + b.Cos\theta \quad and \quad q = b.Sin\theta - a.Cos\theta$$

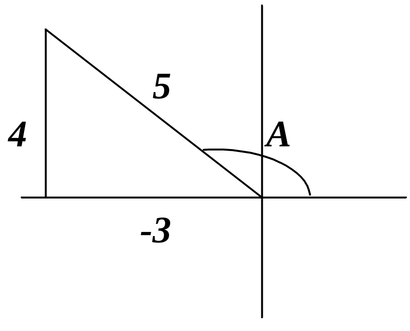
$$p^2 + q^2 = (aSin\theta + bCos\theta)^2 + (bSin\theta - aCos\theta)^2$$

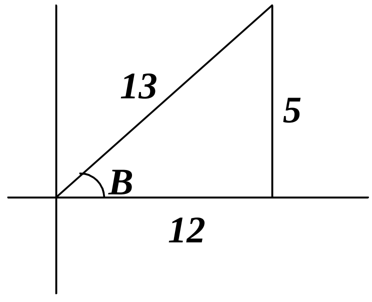
$$= a^2(Sin^2\theta + Cos^2\theta) + b^2(Cos^2\theta + Sin^2\theta) = a^2 + b^2$$

EX-9- If $Sin\ A = 4/5$ and $Cos\ B = 12/13$,where A is obtuse and B is acute . Find , without tables , the values of :

a) $Sin\ (A - B)$, b) $tan\ (A - B)$, c) $tan\ (A + B)$.

Sol. -





$$a)\quad Sin(A-B)=SinA.CosB-CosA.SinB$$
$$=\frac{4}{5}.\frac{12}{13}+\frac{3}{5}.\frac{5}{13}=\frac{63}{65}$$

$$b)\quad tan(A-B)=\frac{tan\,A-tan\,B}{1+tan\,A.tan\,B}$$
$$=\frac{-\frac{4}{3}-\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}}=-\frac{63}{16}$$

$$c)\quad tan(A+B)=\frac{tan\,A+tan\,B}{1-tan\,A.tan\,B}$$
$$=\frac{-\frac{4}{3}+\frac{5}{12}}{1+\frac{4}{3}.\frac{5}{12}}=\frac{33}{56}$$

<u>EX-10</u> **– Prove the following identities:**

$$a)\quad Sin(A+B)+Sin(A-B)=2.SinA.CosB$$
$$b)\quad tan\,A+tan\,B=\frac{Sin(A+B)}{CosA.CosB}$$
$$c)\quad Sec(A+B)=\frac{SecA.SecB.CscA.CscB}{CscA.CscB-SecA.SecB}$$
$$d)\quad \frac{Sin2\theta+Cos2\theta+1}{Sin2\theta-Cos2\theta+1}=Cot\theta$$

Sol.-

$$a)\quad L.H.S. = Sin(A+B)+Sin(A-B)$$
$$= SinA.CosB + CosA.SinB + SinA.CosB - CosA.SinB$$
$$= 2.SinA.CosB = R.H.S.$$

$$b)\quad R.H.S. = \frac{Sin(A+B)}{CosA.CosB} = \frac{SinA.CosB + CosA.SinB}{CosA.CosB}$$
$$= tan\,A + tan\,B = L.H.S.$$

$$c)\quad R.H.S = \frac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB} = \frac{\frac{1}{CosA}\cdot\frac{1}{CosB}\cdot\frac{1}{SinA}\cdot\frac{1}{SinB}}{\frac{1}{SinA}\cdot\frac{1}{SinB} - \frac{1}{CosA}\cdot\frac{1}{CosB}}$$
$$= \frac{1}{CosA.CosB - SinA.SinB} = \frac{1}{Cos(A+B)}$$
$$= Sec(A+B) = L.H.S.$$

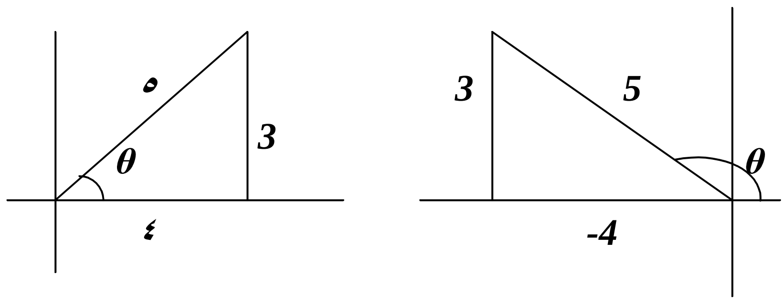
$$d)\quad L.H.S. = \frac{Sin2\theta + Cos2\theta + 1}{Sin2\theta - Cos2\theta + 1} = \frac{2Sin\theta.Cos\theta + (Cos^2\theta - Sin^2\theta) + 1}{2Sin\theta.Cos\theta - (Cos^2\theta - Sin^2\theta) + 1}$$
$$= \frac{2Sin\theta.Cos\theta + 2Cos^2\theta}{2Sin\theta.Cos\theta + 2Sin^2\theta} = \frac{Cos\theta}{Sin\theta} = Cot\theta = R.H.S.$$

<u>EX-11</u> – Find , without using tables , the values of *Sin 2θ* and *Cos 2θ*, when:

a) $Sin\theta = 3/5$, **b)** $Cos\,\theta = 12/13$, **c)** $Sin\,\theta = -\sqrt{3}/2$.

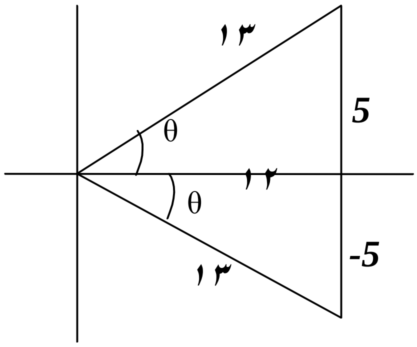
<u>Sol.</u> –

a)



$$Sin2\theta = 2.Sin\theta.Cos\theta = 2.\frac{3}{5}.(\mp\frac{4}{5}) = \mp\frac{24}{25}$$
$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{7}{25}$$

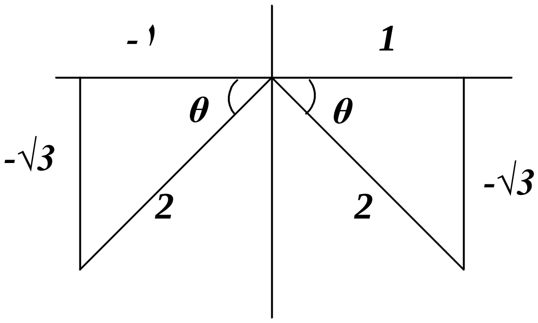
b)



$$Sin2\theta = 2.Sin\theta.Cos\theta = 2(\mp\frac{5}{13}).(\frac{12}{13}) = \mp\frac{120}{169}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\frac{12}{13})^2 - (\mp\frac{5}{13})^2 = \frac{119}{169}$$

c)



$$Sin2\theta = 2Sin\theta.Cos\theta = 2(-\frac{\sqrt{3}}{2}).(\mp\frac{1}{2}) = \pm\frac{\sqrt{3}}{2}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{1}{2})^2 - (-\frac{\sqrt{3}}{2})^2 = -\frac{1}{2}$$

EX-12- **Solve the following equations for values of** θ **from** 0^o **to** 360^o **inclusive:**

a) $Cos\ 2\theta + Cos\ \theta + 1 = 0$, **b)** $4\ tan\ \theta\ .\ tan\ 2\theta = 1$

Sol.-

$$a)\quad Cos2\theta + Cos\theta + 1 = 0 \Rightarrow 2Cos^2\theta - 1 + Cos\theta + 1 = 0$$

$$\Rightarrow Cos(2.Cos\theta + 1) = 0$$

$$either \quad Cos\theta = 0 \Rightarrow \theta = 90^o, 270^o$$

$$or \quad Cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^o, 240^o$$

$$\theta = \{90^o, 120^o, 240^o, 270^o\}$$

$$b)\quad 4.\tan\theta.\tan 2\theta = 1 \Rightarrow 4.\tan\theta.\frac{2\tan\theta}{1-\tan^2\theta} = 1$$

$$\Rightarrow 9\tan^2\theta = 1$$

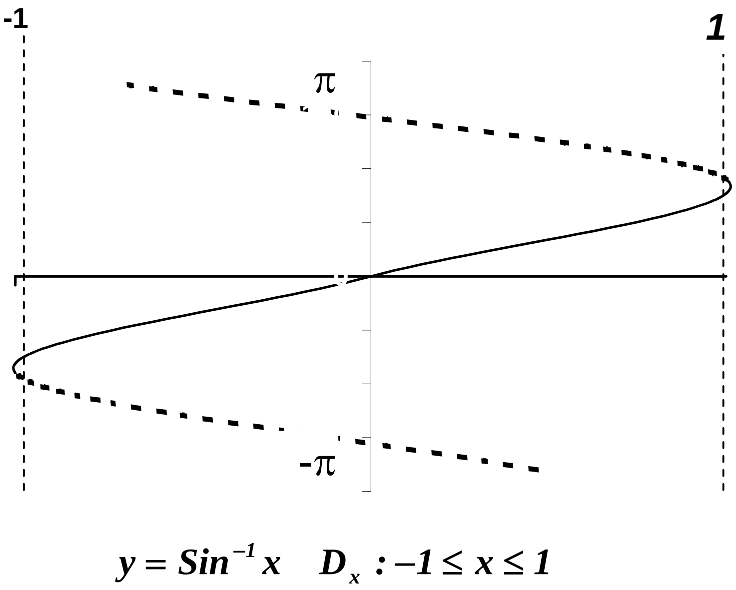
$$either \quad \tan\theta = \frac{1}{3} \Rightarrow \theta = 18.4^o, 198.4^o$$

$$or \quad \tan\theta = -\frac{1}{3} \Rightarrow \theta = 161.6^o, 341.6^o$$

$$\theta = \{18.4^o, 161.6^o, 198.4^o, 341.6^o\}$$

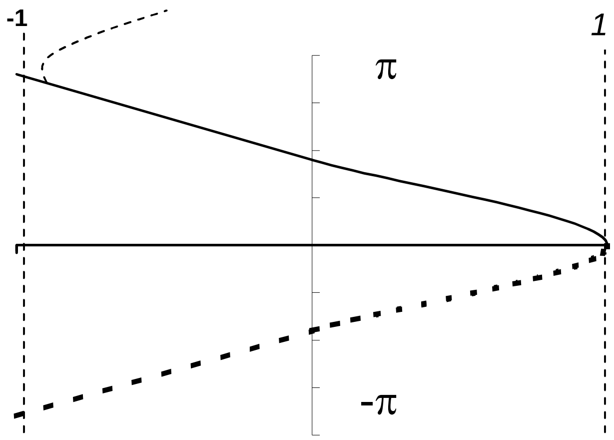
2-3- <u>The inverse trigonometric functions</u> : The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles :

$$y = Sinx \quad \Leftrightarrow \quad x = Sin^{-1}y$$



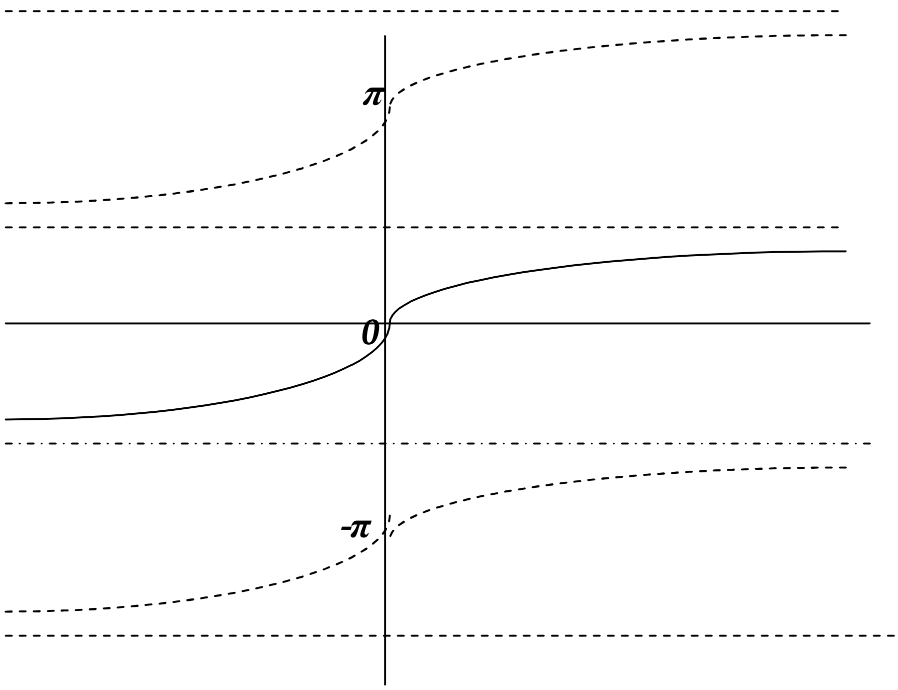
$$y = Sin^{-1}x \quad D_x : -1 \le x \le 1$$

$$R_y : -90 \le y \le 90$$



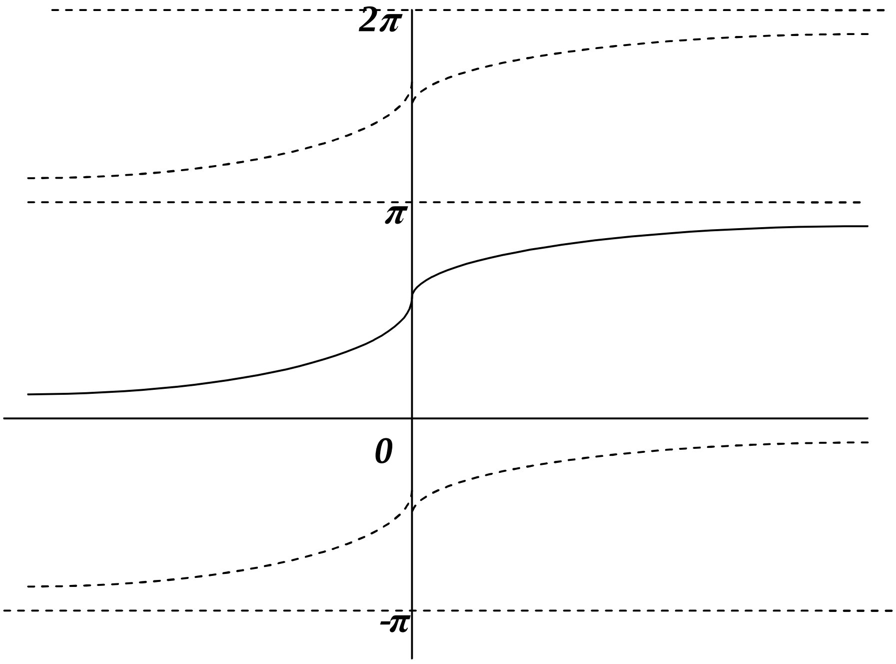
$y = Cos^{-1} x \quad D_x : -1 \le x \le 1$

$R_y : 0 \le y \le 180$

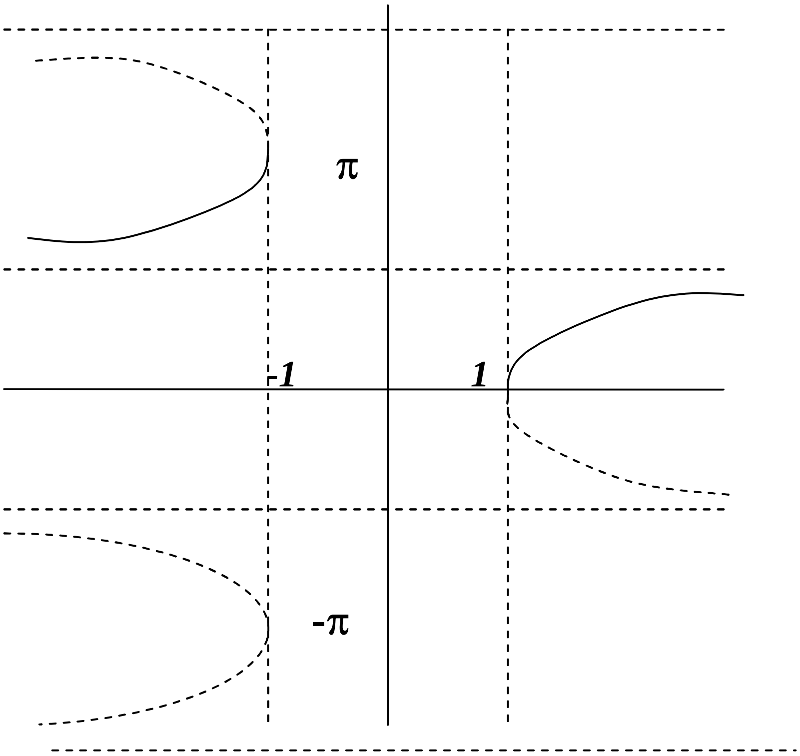


$y = tan^{-1}\ x \quad D_x : \forall x$

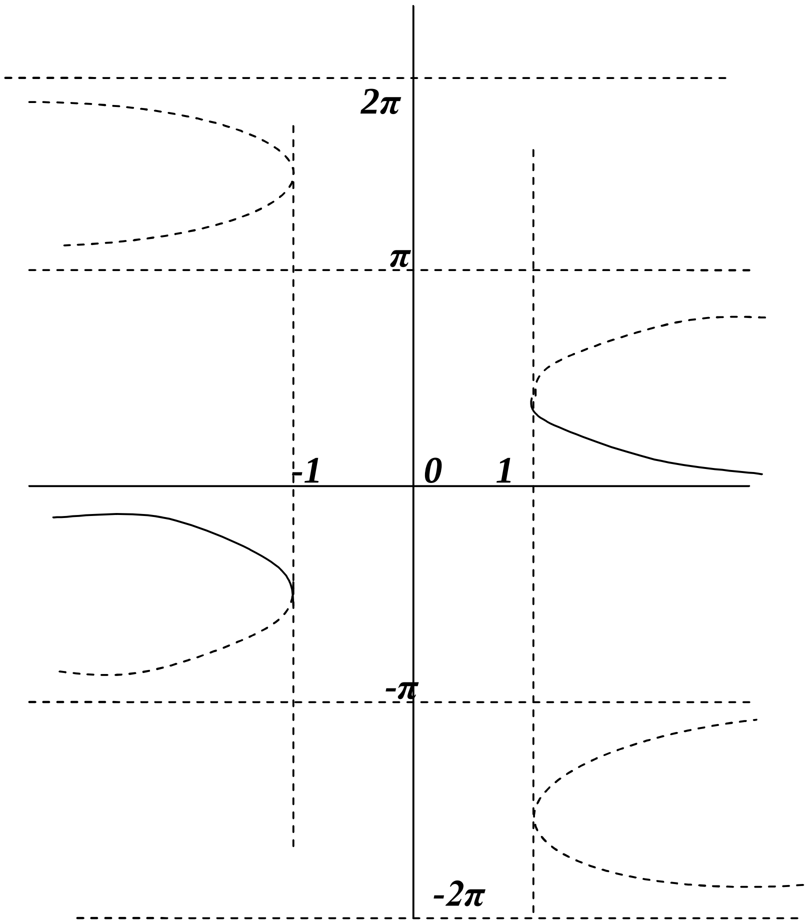
$R_y : -90 \le y \le 90$



$$y = Cot^{-1}x \quad D_x : \forall x$$
$$R_y : 0 \le y \le \pi$$



$$y = Sec^{-1}x \quad D_x : \forall |x| \ge 1$$
$$R_y : 0 \le y \le \pi , y \ne \frac{\pi}{2}$$



$$y = Csc^{-1} x \quad D_x : \forall |x| \geq 1$$

$$R_y : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$$

The following are some properties of the inverse trigonometric functions :

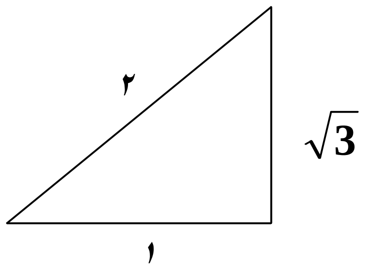
1. $Sin^{-1}(-x) = -Sin^{-1} x$
2. $Cos^{-1}(-x) = \pi - Cos^{-1} x$
3. $Sin^{-1} x + Cos^{-1} x = \frac{\pi}{2}$
4. $tan^{-1}(-x) = -tan^{-1} x$
5. $Cot^{-1} x = \frac{\pi}{2} - tan^{-1} x$
6. $Sec^{-1} x = Cos^{-1} \frac{1}{x}$
7. $Csc^{-1} x = Sin^{-1} \frac{1}{x}$
8. $Sec^{-1}(-x) = \pi - Sec^{-1} x$

and noted that $(Sinx)^{-1} = \frac{1}{Sinx} = Cscx \neq Sin^{-1} x$

EX-13- **Given that** $\alpha = Sin^{-1}\dfrac{\sqrt{3}}{2}$ **, find :**

$Csc\alpha \;\; ,Cos\alpha \;\; ,Sec\alpha \;\; ,tan\,\alpha \;\; ,and \;\; ,Cot\alpha$

Sol.-



$$\alpha = Sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow Sin\alpha = \frac{\sqrt{3}}{2} = \frac{x}{y} \Rightarrow r = \sqrt{4-3} = 1$$

$$Csc\alpha = \frac{2}{\sqrt{3}} \;\; ,Cos\alpha = \frac{1}{2} \;\; ,Sec\alpha = 2 \;\; ,tan\,\alpha = \sqrt{3} \;\; ,Cot\alpha = \frac{1}{\sqrt{3}}$$

EX-14 **– Evaluate the following expressions :**

$$a)\;\; Sec(Cos^{-1}\frac{1}{2}) \quad b)\;\; Sin^{-1}1 - Sin^{-1}(-1) \quad c)\;\; Cos^{-1}(-Sin\frac{\pi}{6})$$

Sol.-

$$a)\;\; Sec(Cos^{-1}\frac{1}{2}) = Sec\frac{\pi}{3} = 2$$

$$b)\;\; Sin^{-1}1 - Sin^{-1}(-1) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$$

$$c)\;\; Cos^{-1}(-Sin\frac{\pi}{6}) = Cos^{-1}(-\frac{1}{2}) = \frac{2}{3}\pi$$

EX-15- **Prove that :**

$$a)\;\; Sec^{-1}x = Cos^{-1}\frac{1}{x} \qquad b)\;\; Sin^{-1}(-x) = -Sin^{-1}x$$

Sol.

$$a) \qquad Let\;\; y = Sec^{-1}x \Rightarrow x = Secy \Rightarrow x = \frac{1}{Cosy}$$

$$\Rightarrow y = Cos^{-1}\frac{1}{x} \Rightarrow Sec^{-1}x = Cos^{-1}\frac{1}{x}$$

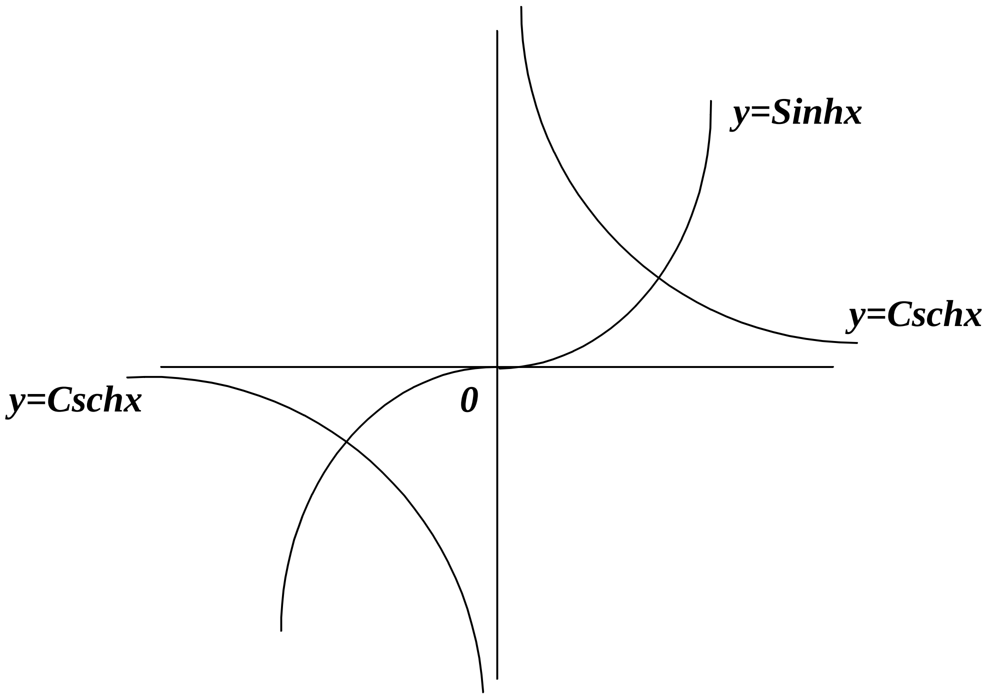
$$b) \quad Let\;\; y = -Sin^{-1}x \Rightarrow x = Sin(-y) \Rightarrow x = -Siny$$

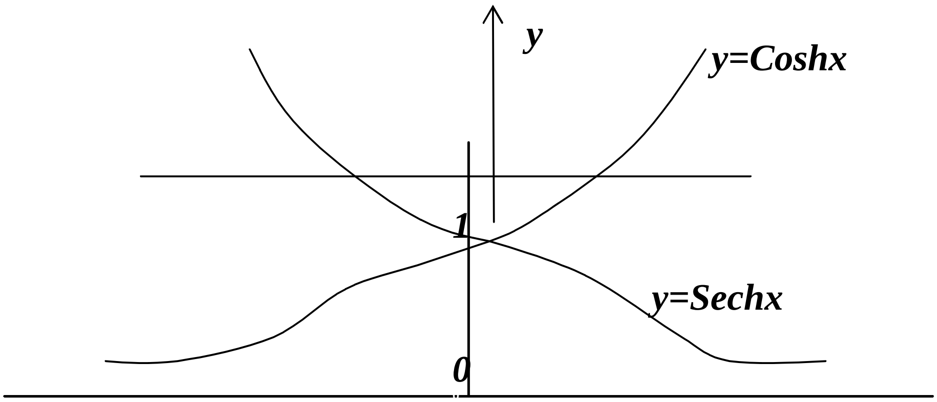
$$\Rightarrow y = Sin^{-1}(-x) \Rightarrow Sin^{-1}(-x) = -Sin^{-1}x$$

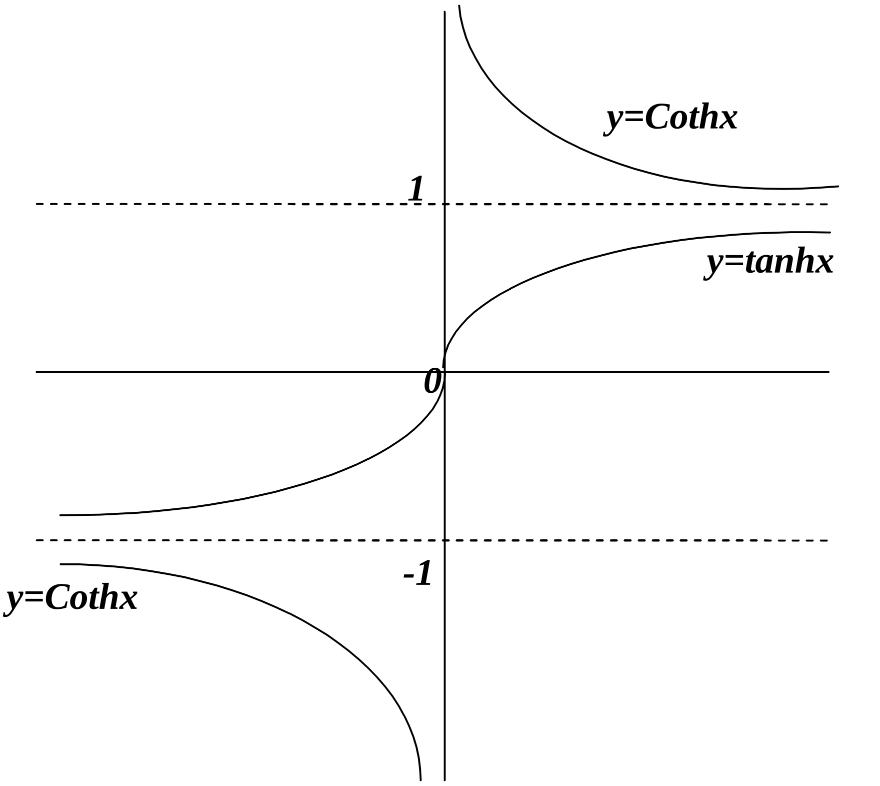
2-4- <u>Hyperbolic functions</u> : Hyperbolic functions are used to describe the motions of waves in elastic solids ; the shapes of electric power lines ; temperature distributions in metal fins that cool pipes …etc.

The hyperbolic sine (Sinh) and hyperbolic cosine (Cosh) are defined by the following equations :

1. $Sinhu = \frac{1}{2}(e^u - e^{-u})$ and $Coshu = \frac{1}{2}(e^u + e^{-u})$
2. $tanh\,u = \frac{Sinhu}{Coshu} = \frac{e^u - e^{-u}}{e^u + e^{-u}}$ and $Cothu = \frac{Coshu}{Sinhu} = \frac{e^u + e^{-u}}{e^u - e^{-u}}$
3. $Sechu = \frac{1}{Coshu} = \frac{2}{e^u + e^{-u}}$ and $Cschu = \frac{1}{Sinhu} = \frac{2}{e^u - e^{-u}}$
4. $Cosh^2u - Sinh^2u = 1$
5. $tanh^2 u + Sech^2u = 1$ and $Coth^2u - Csch^2u = 1$
6. $Coshu + Sinhu = e^u$ and $Coshu - Sinhu = e^{-u}$
7. $Cosh(-u) = Coshu$ and $Sinh(-u) = -Sinhu$
8. $Cosh0 = 1$ and $Sinh0 = 0$
9. $Sinh(x + y) = Sinhx.Coshy + Coshx.Sinhy$
10. $Cosh(x + y) = Coshx.Coshy + Sinhx.Sinhy$
11. $Sinh2x = 2.Sinhx.Coshx$
12. $Cosh2x = Cosh^2 x + Sinh^2 x$
13. $Cosh^2 x = \frac{Cosh2x + 1}{2}$ and $Sinh^2 x = \frac{Cosh2x - 1}{2}$







$y = Sinhx \quad D_x : \forall x \quad and \quad R_y : \forall y$

$y = Coshx \quad D_x : \forall x \quad and \quad R_y : y \geq 1$

$y = \tanh x \quad D_x : \forall x \quad and \quad R_y : -1 \leq y \leq 1$

$y = Cothx \quad D_x : \forall x \neq 0 \quad and \quad R_y : y < -1 \;\; or \;\; y > 1$

$y = Sechx \quad D_x : \forall x \quad and \quad R_y : 0 < y \leq 1$

$y = Cschx \quad D_x : \forall x \neq 0 \quad and \quad R_y : \forall y \neq 0$

EX-16- **Let *tanh u = - 7 / 25* , determine the values of the remaining five hyperbolic functions .**

Sol.-

$$Cothu = \frac{1}{\tanh u} = -\frac{25}{7}$$

$$\tanh^2 u + Sech^2 u = 1 \quad \Rightarrow \frac{49}{625} + Sech^2 u = 1 \Rightarrow Sechu = \frac{24}{25}$$

$$Coshu = \frac{1}{Sechu} = \frac{25}{24}$$

$$\tanh u = \frac{Sinhu}{Coshu} \qquad \Rightarrow -\frac{7}{25} = \frac{Sinhu}{\frac{25}{24}} \Rightarrow Sinhu = -\frac{7}{24}$$

$$Cschu = \frac{1}{Sinhu} = -\frac{24}{7}$$

EX-17- **Rewrite the following expressions in terms of exponentials . Write the final result as simply as you can :**

$a)\ 2Cosh(\ln x) \qquad b)\ \tanh(\ln x)$

$c)\ Cosh5x + Sinh5x \quad d)\ (Sinhx + Coshx)^4$

Sol.-

$$a) \quad 2Cosh(\ln x) = 2.\frac{e^{\ln x} + e^{-\ln x}}{2} = x + \frac{1}{x}$$

$$b) \quad \tanh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}} = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{x^2 - 1}{x^2 + 1}$$

$$c) \quad Cosh5x + Sinh5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$$

$$d) \quad (Sinhx + Coshx)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = e^{4x}$$

EX-18- **Solve the equation for** x **:** $Cosh\ x = Sinh\ x + 1/2$ **.**

Sol. - $Coshx - Sinhx = \frac{1}{2} \Rightarrow e^{-x} = \frac{1}{2} \Rightarrow -x = \ln 1 - \ln 2 \Rightarrow x = \ln 2$

EX-19 – Verify the following identity :

a) *Sinh(u+v)=Sinh u. Cosh v + Cosh u.Sinh v*

b) then verify *Sinh(u-v)=Sinh u. Cosh v - Cosh u.Sinh v*

Sol.-

$$a)\ R.H.S. = Sinhu.Coshv + Coshu.Sinhv$$
$$= \frac{e^{u} - e^{-u}}{2}.\frac{e^{v} + e^{-v}}{2} + \frac{e^{u} + e^{-u}}{2}\frac{e^{v} - e^{-v}}{2}$$
$$= \frac{e^{u+v} - e^{-(u+v)}}{2} = Sinh(u+v) = L.H.S.$$
$$b)\ L.H.S. = Sinh(u+(-v)) = Sinhu.Cosh(-v) + Coshu.Sinh(-v)$$
$$= Sinhu.Coshv - Coshu.Sinhv = R.H.S.$$

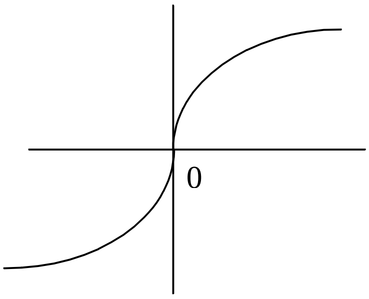
<u>EX-20</u> **– Verify the following identities :**

$$a)\quad Sinhu.Coshv = \frac{1}{2}\left[Sinh(u+v) + Sinh(u-v)\right]$$
$$b)\quad Coshu.Coshv = \frac{1}{2}\left[Cosh(u+v) + Cosh(u-v)\right]$$
$$c)\quad Sinh3u = Sinh^{3}u + 3Cosh^{2}u.Sinhu = 3Sinhu + 4Sinh^{3}u$$
$$d)\quad Sinh^{2}u - Sinh^{2}v = Cosh^{2}u - Cosh^{2}v$$

<u>Sol.</u> **–**

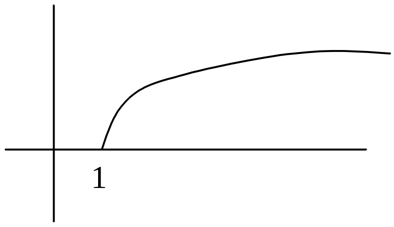
$$a)\ R.H.S. = \frac{1}{2}\left[Sinh(u+v) + Sinh(u-v)\right]$$
$$= \frac{1}{2}\left[Sinhu.Coshv + Coshu.Sinhv + Sinhu.Coshv - Coshu.Sinhv\right]$$
$$= Sinhu.Coshv = L.H.S.$$
$$b)\ R.H.S. = \frac{1}{2}\left[Cosh(u+v) + Cosh(u-v)\right]$$
$$= \frac{1}{2}\left[Coshu.Coshv + Sinhu.Sinhv + Coshu.Coshv - Sinhu.Sinhv\right]$$
$$= Coshu.Coshv = L.H.S.$$
$$c)\ L.H.S. = Sinh(2u+u) = Sinh2u.Coshu + Cosh2u.Sinhu$$
$$= 2Sinhu.Coshu.Coshu + (Cosh^{2}u + Sinh^{2}u).Sinhu$$
$$= 3Sinhu.Cosh^{2}u + Sinh^{3}u = R.H.S.(I)$$
$$= 3Sinhu.(1 + Sinh^{2}u) + Sinh^{3}u = 3Sinhu + 4Sinh^{3}u = R.H.S.(II)$$
$$d)\ L.H.S. = Sinh^{2}u - Sinh^{2}v = Cosh^{2}u - 1 - (Cosh^{2}v - 1)$$
$$= Cosh^{2}u - Cosh^{2}v = R.H.S.$$

2-5- <u>Inverse hyperbolic functions</u> : All hyperbolic functions have inverses that are useful in integration and interesting as differentiable functions in their own right .



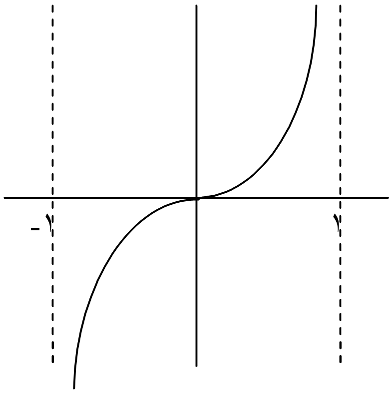
$y = Sinh^{-1} x \quad D_x : \forall x$

$R_y : \forall y$



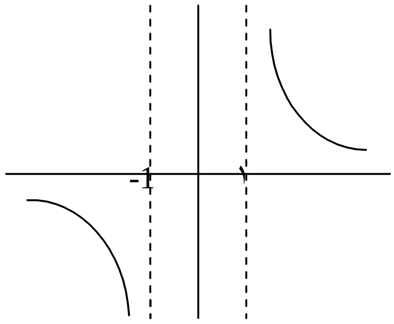
$y = Cosh^{-1} x \quad D_x : \forall x \geq 1$

$R_y : \forall y \geq 0$



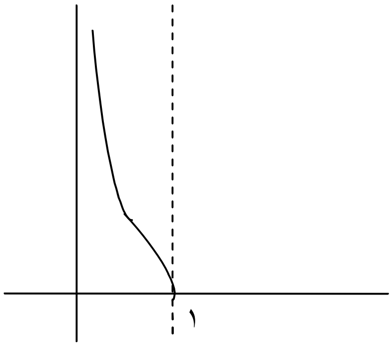
$y = tanh^{-1} x \quad D_x : -1 < x < 1$

$R_y : \forall y$



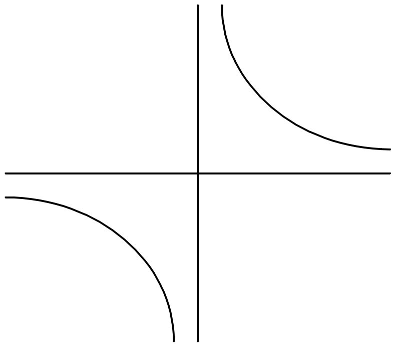
$y = Coth^{-1} x \quad D_x : \forall x < -1 \text{ or } x > 1$

$R_y : \forall y \neq 0$



$y = Sech^{-1} x \quad D_x : 0 < x \leq 1$

$R_y : \forall y \geq 0$



$y = Csch^{-1} x \quad D_x : \forall x \neq 0$

$R_y : \forall y \neq 0$

Some useful identities :

1. $Sinh^{-1}x = ln(\,x + \sqrt{x^2+1}\,)$
2. $Cosh^{-1}x = ln(\,x + \sqrt{x^2-1}\,)$
3. $tanh^{-1}\,x = \frac{1}{2}.ln\left(\frac{1+x}{1-x}\right)$
4. $Coth^{-1}x = \frac{1}{2}.ln\left(\frac{x+1}{x-1}\right) = tanh^{-1}\frac{1}{x}$
5. $Sech^{-1}x = ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = Cosh^{-1}\frac{1}{x}$
6. $Csch^{-1}x = ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right) = Sinh^{-1}\frac{1}{x}$

EX-21 **- Derive the formula :**

$$Sinh^{-1}x = ln(\,x + \sqrt{x^2+1}\,)$$

Sol.-

$$Let\;\; y = Sinh^{-1}x \Rightarrow x = Sinhy = \frac{e^y - e^{-y}}{2} \Rightarrow x = \frac{e^{2y}-1}{2e^y}$$

$$\Rightarrow e^{2y} - 2x.e^y - 1 = 0$$

$$e^y = \frac{2x \mp \sqrt{4x^2+4}}{2} \Rightarrow e^y = x \mp \sqrt{x^2+1}$$

$either\;\; y = ln(\,x - \sqrt{x^2+1}\,) \quad neglected \;\; since\;\; x - \sqrt{x^2+1} < 0$

$or \quad y = ln(\,x + \sqrt{x^2+1}\,)$

Problems – 2

1. **A body of unknown temperature was placed in a room that was held at 30^o *F* . After *10* minutes , the body's temperature was 0^oF , and *20* minutes after the body was placed in the room the body's temperature 15^oF . Use Newton's law of cooling to estimate the body's initial temperature .** *(ans.:-30^oF)*

2. **A pan of warm water 46^oC was put in a refrigerator . Ten minutes later , the water's temperature was 39^oC , *10* minutes after that , it was 33^oC . Use Newton's law of cooling to estimate how cold the refrigerator was ?**
(ans.:-3^oC)

3. **Solve the following equations for values of θ from -180^o to 180^o inclusive:**
   i) $\tan^2\theta + \tan\theta = 0$ ii) $Cot\ \theta = 5\ Cos\ \theta$
   iii) $3\ Cos\ \theta + 2\ Sec\ \theta + 7 = 0$ iv) $Cos^2\theta + Sin\ \theta + 1 = 0$
   (ans.:i)-180,-45,0,135,180; ii)-90,11.5,90,168.5; iii)-109.5,109.5; iv)-90)

4. **Solve the following equations for values of θ from 0^o to 360^o inclusive:**
   i) $3\ Cos\ 2\theta - Sin\ \theta + 2 = 0$ ii) $3\tan\theta = \tan 2\theta$
   iii) $Sin\ 2\theta.\ Cos\ \theta + Sin^2\theta = 1$ iv) $3\ Cot\ 2\theta + Cot\ \theta = 1$
   (ans.:i)56.4,123.6,270; ii)0,30,150,180,210,330,360; iii)30,90,150,270; iv)45,121,225,301)

5. **If $Sin\ \theta = 3/5$, find without using tables the values of :**
   i) $Cos\ \theta$ ii) $\tan\theta$ *(ans.: i) 4/5 ; ii) 3/4)*

6. **Find, without using tables, the values of *Cos x* and *Sin x* , when *Cos 2x* is :**
   a) 1/8 , b) 7/25 , c) -119/169
   $$(ans.:a)\mp\frac{3}{4},\mp\frac{\sqrt{7}}{4};b)\mp\frac{4}{5},\mp\frac{3}{5};c)\mp\frac{5}{13},\mp\frac{12}{13})$$

7. **If $Sin\ A = 3/5$ and Sin $B = 5/13$, where *A* and *B* are acute angles , find without using tables , the values of :**
   a) Sin(A+B) , b) Cos(A+B) , c) Cot(A+B) *(ans.: 56/65; 33/65; 33/56)*

8. **If tan $A = -1/7$ and $\tan B = 3/4$, where *A* is obtuse and *B* is acute , find without using tables the value of $A - B$.** *(ans.: 135)*

9. **Prove the following identities :**

i) $Sec^2\theta + Csc^2\theta = Sec^2\theta . Csc^2\theta$

ii) $Sin^2\theta(1+Sec^2\theta) = Sec^2\theta - Cos^2\theta$

iii) $\dfrac{1+Sin\theta}{1-Sin\theta} = (Sec\theta + tan\theta)^2$

iv) $Sec\theta - Sin\theta = \dfrac{tan^2\theta + Cos^2\theta}{Sec\theta + Sin\theta}$

v) $\dfrac{Cos(A-B) - Cos(A+B)}{Sin(A+B) + Sin(A-B)} = tan B$

vi) $CosB - CosA.Cos(A-B) = SinA.Sin(A-B)$

vii) $tan(A+B+C) = \dfrac{tan A + tan B + tan C - tan A.tan B.tan C}{1 - tan B.tan C - tan C.tan A - tan A.tan B}$

If A,B,C are angles of a triangle, show that:

$tan A + tan B + tan C = tan A.tan B.tan C$

viii) $\dfrac{1}{2}\left[tan(x+h) + tan(x-h)\right] - tan x = \dfrac{tan x.Sin^2 h}{Cos^2 x - Sin^2 h}$

ix) $tan x = \sqrt{\dfrac{1-Cos2x}{1+Cos2x}}$

x) $\dfrac{Sin4A + Sin2A}{Cos4A + Cos2A + 1} = tan 2A$

xi) $Sin^4\theta + Cos^4\theta = \dfrac{1}{4}(Cos4\theta + 3)$

xii) $4Sin^3A.Cos3A + 4Cos^3A.Sin3A = 3Sin4A$

xiii) $tan 3A = \dfrac{3 tan A - tan^3 A}{1 - 3 tan^2 A}$

xiv) $Cos^{-1}(-x) = \pi - Cos^{-1}x$

xv) $Cot^{-1}x = \dfrac{\pi}{2} - tan^{-1}x$

xvi) $Cosh(u+v) = Coshu.Coshv + Sinhu.Sinhv$

and then verify

$Cosh(u-v) = Coshu.Coshv - Sinhu.Sinhv$

xvii) $Coshu.Sinhv = \dfrac{1}{2}\left[Sinh(u+v) - Sinh(u-v)\right]$

xviii) $Sinhu.Sinhv = \dfrac{1}{2}\left[Cosh(u+v) - Cosh(u-v)\right]$

xix) $Cosh3u = Coshu + 4Sinh^2u.Coshu = 4Cosh^3u - 3Coshu$

xx) $(Coshx + Sinhx)^n = Coshnx + Sinhnx$

10. If $u = \dfrac{1+Sin\theta}{Cos\theta}$**, prove that** $\dfrac{1}{u} = \dfrac{1-Sin\theta}{Cos\theta}$ **and deduce formula for** $Sin\theta$, $Cos\theta$, $tan\theta$ **in terms of** *u*. ***(ans.:$(u^2-1)/(u^2+1)$; $2u/(u^2+1)$; $(u^2-1)/(u^2+1)$)***

11. If $Sin(x+\alpha)=2Cos(x-\alpha)$**; prove that :** $tan\,x=\dfrac{2-tan\alpha}{1-2tan\alpha}$ **.**

12. If $Sin(x-\alpha)=Cos(x+\alpha)$**; prove that :** $tan\,x=1$ **.**

13. If $x=Cos\theta+Cos2\theta$ ***and*** $y=Sin\theta+Sin2\theta$ **. Show that :**

i) $x^2-y^2=Cos2\theta+2Cos3\theta+Cos4\theta$

ii) $2xy=Sin2\theta+2Sin3\theta+Sin4\theta$

14. If $Cos2A.Cos2B=Cos2\theta$ **, prove that :**

$Sin^2A.Cos^2B+Cos^2A.Sin^2B=Sin^2\theta$

15. If $S = Sin\ \theta$ **and** $C = Cos\ \theta$ **, simplify :**

i) $\dfrac{S.C}{\sqrt{1-S^2}}$, *ii)* $\dfrac{S.\sqrt{1-S^2}}{C.\sqrt{1-C^2}}$, *iii)* $\dfrac{C}{S}+\dfrac{S}{C}$

(ans.:i) Sinθ; ii)1; iii) Secθ.Cscθ)

16. Eliminate θ **from the following equations :**

i) $x=a.Csc\theta$ *and* $y=b.Sec\theta$

ii) $x=Sin\theta+Cos\theta$ *and* $y=Sin\theta-Cos\theta$

iii) $x=Sin\theta+tan\theta$ *and* $y=Sin\theta-tan\theta$

iv) $x=tan\theta$ *and* $y=tan2\theta$

$$(ans: i)\frac{a^2}{x^2}+\frac{b^2}{y^2}=1;\ ii)x^2+y^2=2;\ iii)\frac{4}{(x+y)^2}-\frac{4}{(x-y)^2}=1;\ iv)y=\frac{2x}{1-x^2})$$

17. In the acute – angled triangle *OPQ* , the altitude *OR* makes angles *A* and *B* with *OP* and *OQ* . Show by means of areas that if *OP=q* , *OQ=p* , *OR=r* : *p.q.Sin(A+B) = q.r.SinA + p.r. SinB.*

18. Given that $\alpha=Sin^{-1}\dfrac{1}{2}$ **, find *Cosα , tanα , Secα , and Cscα.***

$$(ans.:\frac{\sqrt{3}}{2};\frac{1}{\sqrt{3}};\frac{2}{\sqrt{3}};2)$$

19. Evaluate the following expressions :

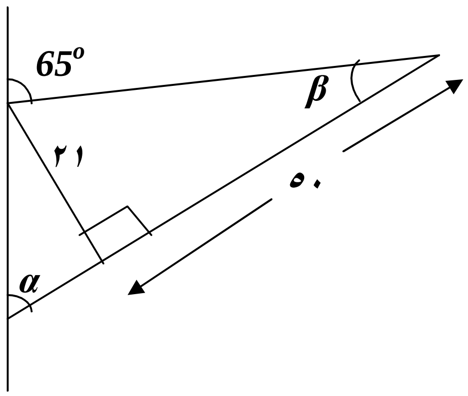
$a)\ Sin(Cos^{-1}\frac{1}{\sqrt{2}})$ $b)\ Csc(Sec^{-1}2)$

$c)\ Cot(Cos^{-1}0)$ $d)\ Sin^{-1}1 - Sin^{-1}(-1)$

$e)\ Cos(Sin^{-1}0.8)$ $f)\ Cos^{-1}(-Sin\frac{\pi}{6})$

$(ans.: 1/\sqrt{2}; 2/\sqrt{3}; 0; \pi; 0.6; 2\pi/3)$

20. Find the angle α in the below graph (Hint : $\alpha+\beta = 65^o$) :



(ans.: 42.2)

21. Let *Sech u = 3/5* , determine the values of the remaining five hyperbolic functions .

$(ans.: Coshu = 5/3; Sinhu = \mp 4/3; tanh\,u = \mp 4/5; Cothu = \mp 5/4; Cschu = \mp 3/4)$

22. Rewrite the following expressions in terms of exponentials , write the final result as simply as you can :

$a)\ Sinh(2.\ln x)$ $b)\ \frac{1}{Coshx - Sinhx}$

$c)\ Cosh3x - Sinh3x$ $d)\ ln(Coshx + Sinhx) + ln(Coshx - Sinhx)$

(ans.:$(x^4-1)/(2x^2)$; e^x ; e^{-3x} ; 0)

23. Solve the equation for *x* ; *tanh x = 3/5* . *(ans.: ln 2)*

24. Show that the distance *r* from the origin *O* to the point *P(Coshu,Sinhu)* on the hyperbola $x^2 - y^2 = 1$ is $r = \sqrt{Cosh2u}$.

25. If θ lies in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and *Sinh x = tan θ* . Show that :

Cosh x = Sec θ , tanh x =Sin θ , Coth x = Csc θ , Csch x = Cot θ , and Sech x = Cos θ .

26. Derive the formula : $tanh^{-1} x = \frac{1}{2} ln\frac{1+x}{1-x}$; $|x| < 1$

27. Find : $\lim_{x\to\infty}\left[Cosh^{-1}x - ln\,x\right]$ **.** *(ans.: ln 2)*

Chapter three
Derivatives

Let $y = f(x)$ be a function of x. If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$$

exists and is finite , we call this limit the derivative of f at x and say that f is differentiable at x .

<u>EX-1</u> – Find the derivative of the function : $f(x) = \frac{1}{\sqrt{2x+3}}$

<u>Sol.</u>:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{2(x+\Delta x)+3}} - \frac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} \cdot \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

<u>Rules of derivatives</u> : Let c and n are constants, u , v and w are differentiable functions of x :

1. $\frac{d}{dx} c = 0$
2. $\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$
3. $\frac{d}{dx} cu = c\frac{du}{dx}$
4. $\frac{d}{dx}(u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$; $\frac{d}{dx}(u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$
5. $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v\frac{du}{dx}$

and $\frac{d}{dx}(u.v.w) = u.v\frac{dw}{dx} + u.w\frac{dv}{dx} + v.w\frac{du}{dx}$

6. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ *where* $v \neq 0$

<u>*EX-2*</u>- Find $\frac{dy}{dx}$ for the following functions :

$a)\ y = (x^2+1)^5$ $b)\ y = [(5-x)(4-2x)]^2$

$c)\ y = (2x^3-3x^2+6x)^{-5}$ $d)\ y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

$e)\ y = \frac{(x^2+x)(x^2-x+1)}{x^3}$ $f)\ y = \frac{x^2-1}{x^2+x-2}$

<u>*Sol.*</u>-

$a)\ \frac{dy}{dx} = 5(x^2+1)^4.2x = 10x(x^2+1)^4$

$b)\ \frac{dy}{dx} = 2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$= 8(5-x)(2-x)(2x-7)$

$c)\ \frac{dy}{dx} = -5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$= -30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

$d)\ y = 12x^{-1} - 4x^{-3} + 3x^{-4} \Rightarrow \frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5}$

$\Rightarrow \frac{dy}{dx} = -\frac{12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$

$e)\ y = \frac{(x+1)(x^2-x+1)}{x^3} \Rightarrow$

$\frac{dy}{dx} = \frac{x^3[(x^2-x+1)+(x+1)(2x-1)] - 3x^2(x+1)(x^2-x+1)}{x^6} = -\frac{3}{x^4}$

$f)\ \frac{dy}{dx} = \frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2} = \frac{x^2-2x+1}{(x^2+x-2)^2}$

The Chain Rule:

1. **Suppose that $h = g_o f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable functions of x whose derivative at each value of x is :**

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

2. **If y is a differentiable function of t and t is differentiable function of x, then y is a differentiable function of x :**

$$y = g(t) \text{ and } t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 **– Use the chain rule to express dy / dx in terms of x and y :**

$$a)\quad y = \frac{t^2}{t^2+1} \qquad and \qquad t = \sqrt{2x+1}$$

$$b)\quad y = \frac{1}{t^2+1} \qquad and \qquad x = \sqrt{4t+1}$$

$$c)\quad y = \left(\frac{t-1}{t+1}\right)^2 \qquad and \qquad x = \frac{1}{t^2} - 1 \qquad at \quad t = 2$$

$$d)\quad y = 1 - \frac{1}{t} \qquad and \qquad t = \frac{1}{1-x} \qquad at \quad x = 2$$

Sol.-

$$a)\quad y = \frac{t^2}{t^2+1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2+1) - 2t.t^2}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$t = (2x+1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

$$b)\quad y=(t^2+1)^{-1}\Rightarrow \frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow \frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$\text{where}\quad x=\sqrt{4t+1}\Rightarrow t=\frac{x^2-1}{4}$$

$$\text{where}\quad y=\frac{1}{t^2+1}\Rightarrow t^2+1=\frac{1}{y}$$

$$c)\quad y=\left(\frac{t-1}{t+1}\right)^2\Rightarrow \frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}=\frac{4(t-1)}{(t+1)^3}$$

$$\Rightarrow\left[\frac{dy}{dt}\right]_{t=2}=\frac{4(2-1)}{(2+1)^3}=\frac{4}{27}$$

$$x=\frac{1}{t^2}-1\Rightarrow\frac{dx}{dt}=-\frac{2}{t^3}\Rightarrow\left[\frac{dx}{dt}\right]_{t=2}=-\frac{2}{2^3}=-\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2}=\left[\frac{dy}{dt}\div\frac{dx}{dt}\right]_{t=2}=\frac{4}{27}\div\left(-\frac{1}{4}\right)=-\frac{16}{27}$$

$$d)\quad t=\frac{1}{1-x}=\frac{1}{1-2}=-1\qquad at\quad x=2$$

$$y=1-\frac{1}{t}\Rightarrow\frac{dy}{dt}=\frac{1}{t^2}\Rightarrow\left[\frac{dy}{dt}\right]_{t=-1}=\frac{1}{(-1)^2}=1$$

$$t=(1-x)^{-1}\Rightarrow\frac{dt}{dx}=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

$$\Rightarrow\left[\frac{dt}{dx}\right]_{x=2}=\frac{1}{(1-2)^2}=1$$

$$\left[\frac{dy}{dx}\right]_{x=2}=\left[\frac{dy}{dt}\right]_{x=2}.\left[\frac{dt}{dx}\right]_{x=2}=1*1=1$$

Higher derivatives **: If a function *y = f(x)* possesses a derivative at every point of some interval , we may form the function *f '(x)* and talk**

about its derivate , if it has one . The procedure is formally identical with that used before , that is :

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\,f'(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$$

if the limit exists .

This derivative is called the second derivative of y with respect to x . It is written in a number of ways , for example,

$$y''\ ,\ f''(x)\ ,\ or\ \frac{d^2 f(x)}{dx^2}\ .$$

In the same manner we may define third and higher derivatives , using similar notations . The *nth* derivative may be written :

$$y^{(n)},\ f^{(n)}(x),\ \frac{d^n y}{dx^n}\ .$$

EX-4- Find all derivatives of the following function :

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol.-

$$\frac{dy}{dx}=9x^2-8x+7 \qquad , \qquad \frac{d^2y}{dx^2}=18x-8$$

$$\frac{d^3y}{dx^3}=18 \qquad , \qquad \frac{d^4y}{dx^4}=0=\frac{d^5y}{dx^5}=....$$

Ex-5 – Find the third derivative of the following function :

$$y=\frac{1}{x}+\sqrt{x^3}$$

Sol.-

$$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{2}{x^3}+\frac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8}x^{-\frac{3}{2}} \qquad \Rightarrow \frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8\sqrt{x^3}}$$

Implicit Differentiation: **If the formula for f is an algebraic combination of powers of x and y . To calculate the derivatives of these implicitly defined functions , we simply differentiate both sides of the defining equation with respect to x .**

***EX-6*- Find** $\frac{dy}{dx}$ **for the following functions:**

$a)\ x^2.y^2 = x^2 + y^2$

$b)\ (x+y)^3 + (x-y)^3 = x^4 + y^4$

$c)\ \frac{x-y}{x-2y} = 2 \ \ at \ \ P(3,1)$

$d)\ xy + 2x - 5y = 2 \ \ at \ \ P(3,2)$

Sol.

$$a)\ x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2y - y}$$

$$b)\ 3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c)\ \frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$$

$$d)\ x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

Exponential functions : **If u is any differentiable function of x , then :**

$$7)\quad \frac{d}{dx}a^u = a^u.\ln a.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}e^u = e^u.\frac{du}{dx}$$